Interaction systems II : the practice of optimal reductions

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PROGRAMME 2

Calcul symbolique,
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The Practice of Optimal Reductions

Les Systèmes d'Interaction II
La Pratique de l'Evaluation Optimale

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Abstract

Lamping's optimal graph reduction technique for the λ-calculus is generalized to a new class of higher order rewriting systems, called Interaction Systems. Interaction Systems provide a nice integration of the functional paradigm with a rich class of data structures (all inductive types), and some basic control flow constructs such as conditionals and (primitive or general) recursion. We describe a uniform and optimal implementation, in Lamping's style, for all these features. The paper is the natural continuation of [3], where we focused on the *theoretical* aspects of optimal reductions in Interaction Systems (family relation, labeling, extraction, ...).

Résumé

La technique d'évaluation optimale de Lamping pour le lambda calcul est étendue à une nouvelle classe de systèmes de réécriture d'ordre supérieur, appelée les Systèmes d'Interaction. Les Systèmes d'Interaction fournissent une bonne intégration du paradigme fonctionel avec une classe trés riche de structures de données (tous le types inductifs), aussi bien qu'avec des constructions de contrôle de flux tel que le conditionnel et la récursion (primitive ou générale). Nous décrivons une implémentation uniforme et optimale dans le style de Lamping, pour toutes ces cas. Ce travail est la suite naturelle de [3], où nous avons étudié les aspects théoriques des réductions optimales dans les Systèmes d'Intéraction (famille des radicaux, etiquetage, extraction ...).

Interaction Systems II
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Abstract

Lamping's optimal graph reduction technique for the λ-calculus is generalized to a new class of higher order rewriting systems, called Interaction Systems. Interaction Systems provide a nice integration of the functional paradigm with a rich class of data structures (all inductive types), and some basic control flow constructs such as conditionals and (primitive or general) recursion. We describe a uniform and optimal implementation, in Lamping's style, for all these features. The paper is the natural continuation of [3], where we focused on the *theoretical* aspects of optimal reductions in Interaction Systems (family relation, labeling, extraction, ...).

1 Introduction

At the end of 70's, Lévy fixed the theoretical performance of what should be considered as an optimal implementation of the λ-calculus. The optimal evaluator should always keep *shared* those redexes in a λ-expression that have a common origin (e.g. that are copies of a same redex). For a long time, no implementation achieved Lévy's performance (see [8] for a quick survey). Only recently, Lamping and Kathail have independently solved the problem [18,13].

Unfortunately, both Lévy's theoretical analysis and the reduction techniques proposed by Lamping and Kathail merely focus on the pure λ-calculus. This is a great limitation in view of an actual implementation, since we must eventually face the problem of extending the language with a wider range of data structures (integers, reals, records, lists, trees, ...) and some basic control flow constructs, such as conditionals, recursion and so on.

Studying the problem of extending Lamping's graph reduction technique to a more expressive language than pure λ-calculus, we discovered Interaction Systems (IS for short). At first glance, the introduction of this new class of rewriting systems, and the

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reason for restricting our analysis to them (IS are just a subclass of Klop's Combinatory Reduction Systems [14]), may look somewhat arbitrary. However, there is a very strong motivation behind our choice, that we would like to explain here.

In [10,11], Gonthier, Abadi and Lévy have provided a remarkable simplification and a more satisfactory theoretical status for Lamping's graph reduction technique. Both results have been obtained by exploiting a nice correspondence between Lamping's optimal implementation of λ-calculus and Girard's Geometry of Interaction for Linear Logic. The main idea behind Linear Logic, is that of making a clear distinction between the *logical* part of the calculus (dealing with the logical connectives) and the *structural* part (dealing with the management of hypotheses). These two different aspects of logical calculi have their obvious correspondent (via the Curry-Howard analogy) inside the λ-calculus. From one side we have the *linear* part (the linear λ-calculus), defining the syntactical operators (the arity, the "binding power", etc.) and their interaction; from the other side we have the *structural* part, taking care of the management (sharing) of resources (i.e. of subexpressions). This can be roughly summarized in the equation

$$\lambda\text{-calculus} = \text{linear } \lambda\text{-calculus} + \text{sharing}.$$

From [10,11], it is clear that Lamping's sharing operators (fan, croissant and square bracket), provide a very abstract framework for an (optimal) implementation of the structural part, which is then interfaced to the linear (or logical) part of the calculus. Therefore it seemed that Lamping-Gonthier's evaluation style could be *smoothly generalized* to a larger class of systems by just replacing the linear λ-calculus with an arbitrary linear calculus. The only proviso to respect was to choose a linear calculus *with a strong logical foundation*, since otherwise we could immediately loose the logical analogy underlying all the previous discussion (in particular, the interface between the linear and the structural part seems to be critical). At this point, we had a natural (and up to our knowledge, unique) candidate for the linear calculus: Lafont's Interaction Nets [15].

Dropping the linearity constraint in Interaction Nets, we just obtain Interaction Systems. In the spirit of the equation above, we could write

$$\text{Interaction Systems} = \text{Interaction Nets} + \text{sharing}.$$

(this is not completely correct, since the nets we consider are *intuitionistic*, and not *classical* as in [15]; however the previous equation provides the main intuition).

Interaction Systems have been introduced in [3] (see also [19]). In the same paper we have also investigated the main theoretical aspects of optimal reductions. In particular, we have defined the notion of *redex family* via a suitable generalization of Lévy's labeling, and we have compared this definition with other well known approaches to the family relation (copy-relation, and extraction process [21]). We remark that, up to our knowledge, this has been the first attempt to generalize the theory of optimal reduction to a super-system of λ-calculus.

The technical preliminaries in [3] (that revealed some unexpected problems, especially with the copy-relation and the extraction process) was eventually aimed to provide the theoretical background for studying implementative issues (we could not avoid it: if we wish to prove that our implementation is optimal, we must eventually start with providing the formal notion of optimality!).

This work is the natural prosecution of [3]. In this paper we define the implementation of Interaction Systems in Lamping-Gonthier's style, and prove its correctness and optimality.

The structure of the paper is the following. In Section 2 we start with introducing the problem of optimal reductions, and optimal sharing in the λ-calculus. Then, we outline Lamping's graph reduction technique, and his approach to correctness (control semantics and read-back). Finally, we briefly discuss the relation between Lamping's implementation and Linear logic [10,11], which provides the guideline for our extension to IS's.

In Section 3 we introduce Interaction Systems as a subclass of Klop's Combinatory Reduction Systems, providing several examples. We also discuss the *intuitionistic* nature of Interaction Systems. This is an essential feature of IS's; in particular, it immediately suggests the design of optimal evaluators, following the ideas in [11].

The formal definition of IS's is in Section 4. In the same section, we also introduce the generalization of Lévy's labeling to IS. Labeling is required to define the notion of redex family (two redex are in a same family if and only if their labels are equal), and thus the notion of optimality. The theoretical aspects of labeling, and its relation with other possible approaches to the notion of family, have been already discussed in [3], so we shall rapidly pass through this topic.

The implementation of IS's in Lamping-Gonthier's style is described in Section 7. Although the generalization of the implementation is not too difficult (for people confident with [11], at least), the correctness and optimality proofs are pretty entangled. The reason is that the corresponding proofs in [10] are based on particular properties of the λ-calculus (in particular, some aspects of the control semantics), which do not generalize to IS's. Roughly, we have been forced to extend Lamping's semantical approach (that looks more general, even if less elegant in the case of the λ-calculus), *but using the simplified set of operators defined in* [10]. In particular we have provided a true *read-back procedure*, which allow to recognize the expressions represented by the sharing graphs. We believe that our proof sheds new light on the correctness aspects of these implementation techniques, even in the restricted case of the λ-calculus.

2 Optimal Reduction

Intuitively, a reduction technique is *optimal* if it is able to profit of all the sharing expressed in the initial term, avoiding useless duplications. Looking for optimal reductions has a great practical interest, since the fact of loosing sharing can cause an exponential

explosion of the time required for reducing the expression. Take, for instance, the term

$$\mathbf{M = n\ 2\ I\ I}$$

where **n** and **2** are Church integers and **I** is the identity. Observe that the rightmost-innermost reduction strategy is obviously linear in n. However, in most of the "standard" implementations for functional languages (such as Combinatory Logic, Supercombinators – as G-Machine and TIM-machine – or Environments machines – as SECD, CAML, Krivine's machine and ZINC – or Continuation Passing Style – as SML –) the evaluation of M grows *exponentially* in n.

Of course, the explanation of this inefficiency should deserve a different analysis for each implementation. However, in this particular case, we may roughly identify the problem in the weak evaluation paradigm adopted by them: since we never reduce inside a lambda, we also loose the possibility of sharing those reductions.

In general, things are not so simple, and we cannot hope to optimize sharing by choosing a suitable evaluation strategy. In particular, Lévy has proved that there are terms, where *every* order of reduction would duplicate work. His favorite example is the following term ([20], p.15):

$$\mathbf{P} \;=\; (\lambda x.\, x\mathbf{I}x \ldots x)\ \lambda y.\, ((\lambda x.\, x \ldots x)\ (y\, a))$$

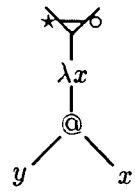
where a is some constant, and the two sequences of x have both length n. **P** has two redexes. If the outermost is reduced first, we eventually create n residuals of the inner one. Conversely, if we start reducing the innermost redex, n copies of $(y\, a)$ are created, and this will duplicate work later on, when **I** is passed as a parameter to y. In conclusion, any reduction strategy is at least linear in n whilst an optimal compiler should be able to get (a suitable representation of) the normal form of **P** in constant time!

2.1 Lamping's solution

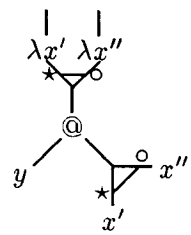
Consider Wadsworth's graph rewriting technique for the evaluation of functional expressions. Every time you have a redex $r = (\lambda x.M)N$ you should start with duplicating the functional part $F = \lambda x.M$. Indeed, F could be shared by other terms, and since we are going to instantiate it, we must eventually work on a new copy (see for instance [12]). If F contains a redex, this will be duplicated as well. The fact of reducing F first, does not help that much. The "redex" inside F could be only a "virtual" one (see [22,5,7] for the formal notion of "virtual" redex). Suppose for instance to have in F a subterm like (yP), where y is bound externally to r. The subterm (yP) is not a redex, but its duplication can be as useless and expensive as the duplication of an actual redex. Moreover, by the considerations in the previous section, the problem cannot be simply solved by the choice of a suitable reduction strategy.

Lamping [17,18] proposed to duplicate F in a sort of "lazy" way, by propagating a duplication operator (a fan) along the graph structure of F, and stopping this propagation at suitable positions (typically, just before the applications in F). Suppose for

instance to have the following configuration, where a duplication operator is applied to $M = \lambda x.(yx)$.

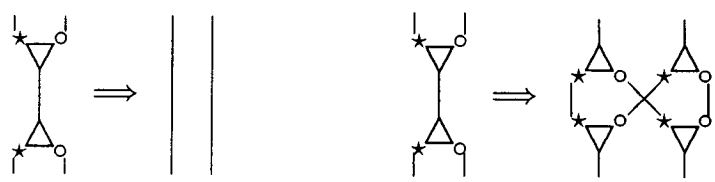


The duplication is done step by step, following the connected structure of M. The first syntactical form traversed by the fan is the binder for x. Note that the duplication of the binder implies the duplication of the bound variable (otherwise we would not know to which of the two binders we should refer). So we obtain the following term.



This term is in normal form. No one of the two fan operators can be propagated any further, until y will be instantiated to a functional term, and the corresponding redex will be fired.

Suppose to replace y by the identity. After firing the redex, we get a graph where two fans meet "face to face". Two reductions seem to be possible, now (but in this case, only the first one is correct):



By the left rule, the effacement of the two fans "completes" the duplication; this rule should be only applied when the two fans belong to the *same* "duplication process". In all the other cases, fans should "mutually cross" each other, according to the right rule, above.

The ambiguity whether applying the left or the right rule is solved by looking at the *sharing-level* of each fan (an integer, denoting the "duplication process" it belongs to). When two fans meet face to face, they will reduce according to the first rule above if they belong to a same level, and according to the second rule, if their levels are different.

Matters are furtherly complicated by the fact that levels may change dynamically during the computation. The menagement of levels requires the introduction of a suitable set of control operators (brackets and croissants) delimiting levels along the computation.

2.2 Context Semantics and Read-back

In Figure 1, we have depicted a typical example of graph (in normal form) which can be obtained as a result of a reduction in Lamping's system [18]. This term can be obtained, for instance, by reducing $\lambda f.\lambda x.(\lambda g.(g(gx))\lambda y.(fy))$. If the implementation is correct, the graph in Figure 1 should thus represent the term $\lambda f.\lambda x.(f(fx))$, but how could we retrieve this information from the graph?

This problem is known as *read-back*, and it is the foremost problem, in proving the correctness of the implementation.

Let us try to explain the general idea by "reading back" the graph in Figure 1. As usual with graphical representations, we start from the root and try to recover the expression by traveling along the graph. The two first nodes we meet are two λ. So far, so easy: the original expression must have the form $\lambda f.\lambda x.X$. The next form we meet is a fan node. In particular, we enter the fan from its $\star$-branch. This information must be recorded: it is a semantical component of a *context* associated with the path we are following in the term. Continue the trip, exiting from the principal port of the fan (fan nodes are discarded by the read-back procedure: they do not appear in the syntactic expression). We reach a @-node, thus X has the form $@(X_1, X_2)$. The subexpression X_1 is found immediately in the left branch of the lower @: it is the

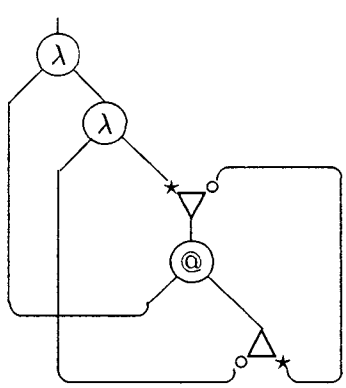


Figure 1: The sharing graph representation of a λ-expression

variable f. The expression X_2 is less obvious. Traveling along the right branch of @ we reach a fan-out node. What branch should we choose? We use the context semantic, to solve the problem. Remember that the last time we traversed a fan-in, we entered from a $\star$-branch (the $\star$ is the top level information in the current context). So we decide to follow the $\star$-branch of the fan out (at the same time, the $\star$ is discharged form the context). Traveling along this branch, we come back to the first fan. In this case, we enter from the o, which becomes the new top level information in the context. Next we find again the application, so X_2 must have the form $@(X_3, X_4)$. As above, we immediately recognize X_3 as the variable f. We have still to determine X_4. Since the top level control information is now o, this time we must follow the o-branch of the

fan-out, finding the variable x.

Summing up, the distinction between branches of fan-nodes is essential to recover correctly the original λ-expression. However it is not powerful enough to solve any possible situation that could rise computing λ-expressions (e.g. matching correctly fan-ins and fan-outs). Therefore the introduction of other control operators as brackets and croissants (see [18]). In particular, the *control information* must be structured in different levels. The semantical effect of control operators is that of creating, discharging, freezing and unfreezing levels.

2.3 Optimal Reduction and Linear Logic

There is a nice relation between Lamping's rewriting system and Girard's Linear Logic which has been pointed out in [10,11]. The general idea is pretty simple: the "level" of each fan is related to the number of nested boxes in the Proof Net representation of the λ-term. Moreover, the control operators mark the extent of each box, implicitly defining their scope. From this point of view, we may consider Lamping's system as a local implementation of the (global) operation of duplication over boxes, in Linear Logic.

Exploting this relation has led to a simpler rewriting system (only 12 rules), together with a much more elegant, *logical* foundation of Lamping's work.

In order to take advantage of this relation, any attempt of generalizing Lamping approach to more powerful rewriting systems, should presuppose some "logical nature" of the calculus. This is the main reason for restricting the analysis to Interaction Systems.

As a matter of fact, the intuitionistic flavour of Interaction Systems directly suggests the design of its optimal evaluator. In particular, let L be the intuitionistic logic associated with an IS (see Section 3.3). We can define a "linear logic" version of L. That is, we may define a new system L_{LL} by just replacing the structural part of L with its "counterpart" in Linear Logic (i.e. giving to weakening and contraction a "logical status" by means of the operators *why not* and *of course*). Then, L can be embedded into L_{LL} in essentially the same way that Intuitionistic Logic is embedded into (Intuitionistic) Linear Logic. Last, by using the optimal implementation of *boxes* defined in [11], we get an optimal implementation of the original IS's. Although we shall not explicitly provide the definition of L_{LL}, the reader should keep in mind the previous methodological assumption, in order to understand the translation of terms using Lamping-Gonthier's operators.

Let us just make here some remarks about this translation, which are not subsumed by the previous discussion.

First of all, we work in an untyped setting. So, as in the case of the λ-calculus, we must add to the "logical" system some implicit type-isomorphism taking care of this fact. For coding λ-calculus in proof nets of Linear Logic, there are two possible

"standard" solutions, respectively based on the type-isomorphism $D \cong !(D \multimap D)$ and $D \cong (!D) \multimap D$. The first one is adopted in [18,10], since it was closer to Lamping's original approach. The second one, is suggested by the "traditional" embedding of intuitionistic implication by means of linear implication [9]. We will follow the latter, since it is closer to the logical perspective (but the former would work as well). As a consequence, our implementation of the λ-calculus will be slightly different from [10].

A second point which requires some care is the implementation of rewriting rules (cuts), since right hand sides (shortened into rhs's) must be "linearized" w.r.t. the metavariables. This problem does not appear in the λ-calculus because β-reduction is already linear in its metavariables. In IS's, in general, this is not true: metavariables can be erased and/or duplicated by each rule. The case of erasing is not particularly problematic, but duplication is more subtle, since it could affect optimality. For the sake of clarity, we shall translate each rule in several steps. We first apply a *linearization* procedure; then translate it according to the standard paradigm in [10] for representing boxes; finally we partially evaluate the rule, eliminating every redundancy which has been possibly introduced in the previous steps.

Finally, we emphasize that the rewriting system describing the optimal implementation of an IS is itself an Interaction Net. This is particularly nice: we started with IN, added sharing to get IS, and implemented (optimally) this sharing inside the original systems. Since IN's get rid of variable names and implement rewriting systems in a symbolic way, our work generalizes some recent results of Burroni [6] and Lafont [16] to higher order rewriting systems.

As much as possible we shall avoid any reference to Interaction Nets and Linear Logic. However [11] is a prerequisite for a deep understanding of the evaluators.

3 Interaction Systems

As we mentioned in the Introduction, Interaction Systems should be correctly understood as the intuitionistic generalization of Lafont's Interaction Nets [15]. The relation between Interaction Systems and Interaction Nets has been deeply investigated in [3]. We shall follow here a different approach (also for avoiding repetition), explaining Interaction Systems as a subclass of Klop's [14] Combinatory Reduction Systems (CRS, for short). In particular, we shall see that Interaction Systems are just *that* subclass of CRS where the Curry-Howard (Proof as Proposition) analogy can still be applied. This will allow us to stress the *intuitionistic* nature of Interaction Systems.

3.1 IS and CRS

It is not our intention to provide the formal definition of Combinatory Reduction Systems, here: we shall just hint the main ideas (the reader is referred to [1,14] for more details).

Combinatory Reduction Systems (CRS) are a higher order generalization of Term Rewriting Systems, where each form of the syntax may work as a binder. The main consequence is that in the right hand side of a rewriting rule can possibly appear *substitutions*, which are defined (in the obvious way) at a meta-level, as in the λ-calculus.

Since each syntactical form $\mathbf{f}$ can act as binder, its *arity* cannot be just an integer n, expressing the number of its arguments, as at the first order level. Indeed, for any argument M, we must also specify the number of variables bound by $\mathbf{f}$ in M. So the arity of a form $\mathbf{f}$ will be a sequence $k_1 \ldots k_n$, where n is the number of arguments, and k_i the number of variables which are bound by $\mathbf{f}$ inside its i-th argument.

The terms of the language are then defined out of forms and variables in the obvious inductive way, according to the arity of the forms.

The next step is to introduce a notion of meta-variable, ranging over terms. This is *very much* like in the λ-calculus; the only care is in defining the proper *arity* for each metavariable X, that is, roughly, the number of distinguished free variables in X (names, not occurrences) which are "accessible" for substitution (in the λ-calculus, this is always 1). So, if the arity of X is k, we may apply to X k-ary substitution $X[^{M_1}/_{x_1}, \ldots, ^{M_k}/_{x_k}]$.

Terms containing metavariables (and substitutions) are called metaterms. A *reduction rule* is any pair (L, R) of metaterms such that:

1. the root form of L is a form;
2. L and R are closed;
3. all the metavariables in R occur already in L;
4. the metavariables in L occur only at leaf-positions in (the abstract syntax tree of) L (in particular, no substitution is applied to them).

If no metavariable occur twice in L, the reduction rule is called *left-linear*. A CRS where all reduction rules are left linear, and without critical pairs, is called *regular*.

Example 3.1 The recursion operator μ, is a form with arity 1: it binds exactly one variable inside its unique argument. The rewriting rule for μ is expressed as follows:

$$\mu(\langle x \rangle . X) \rightarrow X[^{\mu(\langle x \rangle . X)}/_x]$$

where X is a metavariable of arity 1. ■

Interaction Systems are obtained from CRS's by imposing the following constraints (see section 4 for the formal definition):

- in the signature, we have a bipartition of forms in *constructors* and *destructors*. Constructors have arbitrary arities, while the arity of a destructor must have a leading 0. In other words, a destructor cannot bind variables in its first argument.

The reason for this restriction is that the first argument position of a destructor **d** is the (unique) place where it may *interact* with a constructor, and **d** cannot bind variables in the argument is interacting with. We will show that this restriction has a strong logical motivation;

- in the rewriting rules, we just impose a restriction on the shape of the left hand side L, that must look as follow:

$$\mathbf{d}(\mathbf{c}(\vec{x}^1_{k_1}.X_1,\cdots,\vec{x}^m_{k_m}.X_m),\cdots,\vec{x}^n_{k_n}.X_n)$$

where $i \neq j$ implies $X_i \neq X_j$ (*left linearity*). The arity of **d** is $0k_{m+1}\cdots k_n$ and that of **c** is $k_1\cdots k_m$.

In other words, every rewriting rule is defined by the *interaction* of a destructor **d** with a constructor **c** (that is assumed to be unique, i.e. there exists at most one rewriting rule for every pair **d**-**c**).

The previous constraints may look very restrictive and somewhat arbitrary. We shall try to answer to these objections in the following subsections. We shall start with providing several examples of Interaction Systems, in order to show their expressive power. Then we shall discuss the *intuitionistic* nature of Interaction Systems, that motivated their introduction.

3.2 Examples

The most typical example of Interaction System is λ-calculus.

Example 3.2 (**The λ-calculus**) The application @ is a destructor of arity 00, and λ is a constructor of arity 1. The only rewriting rule is β-reduction:

$$@(\lambda(\langle x\rangle.X),Y) \;\rightarrow\; X[^Y/_x]. \qquad \blacksquare$$

An Interaction System where no form is a binder is called *discrete*. Discrete Interaction Systems are obviously a special case of Term Rewriting Systems. The signature of the system is a first order signature Σ partitioned in two classes: the constructors (ranged over by **c**) and the destructors (ranged over by **d**). The rewriting rules have the following general shape:

$$\mathbf{d}(\mathbf{c}(X_1,\cdots,X_m),\cdots,X_n) \;\rightarrow\; H$$

where H is a terms built up with forms and variables in $\{X_1,\cdots,X_n\}$.

A lot of interesting IS's are discrete.

Example 3.3 A typical example of discrete Interaction System is *Primitive Recursion*. There are only two constructors 0 and succ. Composition of two functions $\mathbf{f}(-)$ and $\mathbf{g}(-)$ is obviously expressed as $\mathbf{f}(\mathbf{g}(-))$. The primitive recursion scheme has already the correct IS-shape:

$$\mathbf{d}(0, X) \rightarrow \mathbf{h}(X)$$
$$\mathbf{d}(\mathtt{succ}(X), Y) \rightarrow \mathbf{f}(X, Y, \mathbf{g}(X, Y))$$

For instance, we may define

$$\mathtt{add}(0, X) \rightarrow X \qquad \mathtt{mult}(0, X) \rightarrow 0$$
$$\mathtt{add}(\mathtt{succ}(X), Y) \rightarrow \mathtt{succ}(\mathtt{add}(X, Y)) \qquad \mathtt{mult}(\mathtt{succ}(X), Y) \rightarrow \mathtt{add}(Y, \mathtt{mult}(X, Y)) \quad \blacksquare$$

In a similar way, we may define all inductive types (booleans, lists, trees, and so on).

Example 3.4 Booleans are defined by two constructors **T** and **F** of arity ε (two constants). Then you may add your favorite destructors. A typical example is the *if-then-else* operator $\natural$, of arity 000. The rules for the conditional are described by the following obvious interactions between $\natural$ and **T** or **F**:

$$\natural(\mathbf{T}, X, Y) \rightarrow X$$
$$\natural(\mathbf{F}, X, Y) \rightarrow Y \quad \blacksquare$$

Example 3.5 IS's may be infinite. For instance, when doing arithmetics, we would not like to use the unary notation based on 0 and **succ**. A simple solution is to consider each integer n as a distinguished constructor **n**. Then, we may reasonably define arithmetical operations in constant time. The only problem is the local sequentiality constraint, that imposes interaction on a distinguished port of the form (however, this is what occurs in practice, on a sequential machine). For instance, we may define

$$\mathtt{add}(\mathtt{m}, X) \rightarrow \mathtt{add}_{\mathtt{m}}(X) \qquad \mathtt{add}_{\mathtt{m}}(\mathtt{n}) \rightarrow \mathtt{k}$$

where $\mathtt{k} = \mathtt{n}+\mathtt{m}$. Note that we have an infinite number of forms, and also an infinite number of rewriting rules. $\blacksquare$

Let us remark a trivial but important property of Discrete Interaction Systems, that could motivate the bipartition of forms into constructors and destructors.

Proposition 3.6 If in a Discrete Interaction System we have a rewriting rule for every pair **d**-**c**, then all closed terms in normal form may only contain constructors. $\blacksquare$

In other words, constructors may be used to define the "abstract data type", whose definition is then unaffected by the introduction of new destructors (provided that the definition of each destructor is complete on the data).

Another interesting property of discrete Interaction Systems is that they have trivial optimal implementations. Indeed, since we do not have bindings, they can be represented as acyclic graphs. This means that we never introduce fan-outs during the reduction and so we do not even need control operators (brackets and croissants) to match fan-ins and fan-outs (see [6,16], where the implementations are optimal).

There is an important point to be understood here. As just remarked, it is trivial to provide an optimal implementation of Discrete Interaction Systems. Since they are Turing-complete (there is a trivial encoding of Combinatory Logic), one may wonder what is the interest to consider higher order systems, where the correct handling of sharing becomes much more difficult. For instance, in the case of λ-calculus, we may compile a λ-term M in a term M' of Combinatory Logic, and then reduce M' in an optimal way. The problem is that the optimal reduction of M' has nothing to do with optimality in the λ-calculus! (see [3]).

Example 3.7 Let us finally consider another example out of the discrete case: the recursion operator μ. This is a bit problematic, since IS's are based on a principle of *binay* interaction and in the case of μ we just have a sort of "unary" interaction.

There are two "standard" ways to force a binary interaction for these kind of operators. The first consists in considering them as constructors and to introduce dummy destructors of arity 0 interacting with them. Thus, in the case of μ, we take a destructor $\mathbf{d}_\mu$ and the rewriting rule becomes:

$$\mathbf{d}_\mu(\mu(\langle x\rangle.\, X)) \;\rightarrow\; X[{}^{\mathbf{d}_\mu(\mu(\langle x\rangle.X))}/_x]$$

The dual way consists in considering operators interacting unarily as destructors, and require the existence of "dual" dummy constructors of arity ε. In the case of μ, the dummy constructor is $\mathbf{c}_\mu$ and the rewriting rule becomes:

$$\mu(\mathbf{c}_\mu,\, \langle x\rangle.\, X) \;\rightarrow\; X[{}^{\mu(\mathbf{c}_\mu,\langle x\rangle.X)}/_x] \qquad \blacksquare$$

3.3 The Intuitionistic Nature of IS

We shall now defend our claim that Interaction Systems are the subsystem of Klop's CRS, where the Curry-Howard analogy "still makes sense". We shall do that by stressing the intuitionistic nature of Interaction Systems: constructors and destructors respectively correspond to right and left introduction rules, interaction is cut, and computation is cut-elimination.

3.3.1 From Intuitionistic Systems to IS's ...

An Intuitionistic System, in a *sequent calculus* presentation (*à la* Gentzen), consists of expressions, named *sequents*, whose shape is $A_1, \cdots A_n \vdash B$ where A_i and B are formulas and the comma in the left side of the entail is interpreted as conjunction. Inference rules are partitioned into three groups (in order to emphasize the relationships with IS's, we write rules by assigning terms to proofs):

(Structural Rules)

$$(\textit{Exchange}) \qquad \frac{\Gamma,\, x : A,\, y : B,\, \Delta \vdash t : C}{\Gamma,\, y : B,\, x : A,\, \Delta \vdash t : C}$$

$$(\textit{Contraction})\quad \frac{\Gamma, x:A, y:A \vdash t:C}{\Gamma, z:A, \Delta \vdash t[^{z}/_{x},^{z}/_{y}]:C} \qquad (\textit{Weakening})\quad \frac{\Gamma \vdash t:C}{\Gamma, z:A \vdash t:C}$$

(Identity Group)

$$(\textit{Identity})\quad \frac{}{x:A \vdash x:A} \qquad (\textit{Cut})\quad \frac{\Gamma \vdash t:A \qquad \Delta, x:A \vdash t':B}{\Gamma, \Delta \vdash t'[^{t}/_{x}]:B}$$

(Logical Rules) These are the "peculiar" operations of the systems. Standard operations are implication, conjunction, etc.: in the following we will discuss several examples. What is important to remark here is that logical rules are split into two groups: those introducing the logical connective "on the left" of the sequent and those introducing symbols "on the right". The former ones are called *destructors*; the latter ones are named *constructors*. The shape of these rules will be:

$$\frac{\Gamma_1, \vec{x}^1 : \vec{A}^1 \vdash t_1 : B_1 \quad \cdots \quad \Gamma_m, \vec{x}^m : \vec{A}^m \vdash t_m : B_m \quad \Delta, z:C \vdash t:D}{\Gamma_1, \cdots, \Gamma_m, \Delta, y : \mathtt{T_d}(\vec{A}^1, B_1, \cdots, \vec{A}^m, B_m, C) \vdash t[^{\mathtt{d}(y, \vec{x}^1.t_1, \cdots, \vec{x}^m.t_m)}/_{z}] : D}$$

for destructors and

$$\frac{\Gamma_1, \vec{x}^1 : \vec{A}^1 \vdash t_1 : B_1 \quad \cdots \quad \Gamma_n, \vec{x}^n : \vec{A}^n \vdash t_n : B_n}{\Gamma_1, \cdots, \Gamma_n \vdash \mathtt{c}(\vec{x}^1.t_1, \cdots, \vec{x}^n.t_n) : \mathtt{T_c}(\vec{A}^1, B_1, \cdots, \vec{A}^n, B_n)}$$

for constructors. Above $\mathtt{T_d}(\vec{A}^1, B_1, \cdots, \vec{A}^m, B_m, C)$ and $\mathtt{T_c}(\vec{A}^1, B_1, \cdots, \vec{A}^n, B_n)$ are types built up by means of the types they take as argument and they are equal, provided they correspond to the same logical operator. The unique proviso is that no commitment is done about the contexts Γ_i, that is they are assumed pairwise different. More precisely we are generalizing the so-called *multiplicative* connectives (see [19], pg. 47 for a discussion about additives).

A standard example is implication, that gives the expressions of typed λ-calculus:

$$(\rightarrow \textit{left})\quad \frac{\Delta \vdash t:A \qquad z:B, \Gamma \vdash t':C}{\Delta, y:A \rightarrow B, \Gamma \vdash t'[^{@(y,t)}/_{z}]:C} \qquad (\rightarrow \textit{right})\quad \frac{\Delta, x:A \vdash t:B}{\Delta \vdash \lambda(\langle x \rangle.t) \ : \ A \rightarrow B}$$

An easy consequence of the above construction is that every proof of an Intuitionistic System can be described by an IS-expression. Note in particular that destructors (as the application @) cannot bind at the level of the variable y (the *principal port* of destructors, in Lafont's terminology) since it is a newly added hypothesis and it is found in the lhs of the final sequent. This is the reason why, in the concrete syntax for IS's, we have assumed that destructors have arity 0 at the first argument.

An important theorem of sequent calculus is that stating the redundancy of cut-rules, i.e. every proof with instances of the cut-rule can be turned into an equivalent

one without cuts. From the operational point of view this gives dynamics, because it guarantees the logical soundness of *rewriting* a proof into another one. These rewritings must be specified *a priori*: a proof ending into a cut must be remade into another one by means of some mechanism that is characteristic of that cut.

In order to ensure the possibility of eliminating *all* cuts, the cut-elimination (term rewriting) process must *terminate*. In general, this property does not hold in IS's. We just have a general correspondence between IS's and systems with an *intuitionistic nature*, but only *a posteriori* we may actually check if a particular system enjoys good "logical" properties (cut-elimination, subformula property, ...).

Obviously, we could proceed the other way round, imposing some sufficient conditions on IS's (typing, first of all) to establish a tighter relation with logic. This is surely an interesting subject, but it is out of the scope of the present paper. So, in the following, we shall merely focus on the *dynamic* aspects of cut-elimination, without worrying with termination.

Let us recall what happens in intuitionistic logic when we try to eliminate a cut between the instances of the destructor and the constructor of the implication. Here is the typical situation:

$$\dfrac{\dfrac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda(\langle x \rangle . t) : A \to B} \qquad \dfrac{\Delta \vdash t' : A \qquad y : B, \Theta \vdash t'' : C}{\Delta, z : A \to B, \Theta \vdash t''[^{@(z,t')}/_{y}] : C}}{\Gamma, \Delta, \Theta \vdash t''[^{@(z,t')}/_{y}][^{\lambda(\langle x \rangle . t)}/_{z}] : C}$$

The elimination of the above cut consists in introducing two cuts of lesser grade. The rewritten proof is:

$$\dfrac{\Delta \vdash t' : A \qquad \dfrac{\Gamma, x : A \vdash t : B \qquad y : B, \Theta \vdash t'' : C}{\Gamma, x : A, \Theta \vdash t''[^{t}/_{y}] : C}}{\Gamma, \Delta, \Theta \vdash t''[^{t}/_{y}][^{t'}/_{x}] : C}$$

Note that this meta-operation on proofs induces a rewriting rule in the underlieing IS, which is, in this case, β-reduction. Indeed, it is easy to check that the proofs $t'[^{@(z,t)}/_{y}][^{\lambda(\langle x \rangle . t)}/_{z}]$ and $t''[^{t}/_{y}][^{t'}/_{x}]$ can be proved equal via β-reduction.

Back to the discussion about a generic cut-elimination, we have to understand what kind of rewriting this process performs. Foremost, there are several kinds of cuts: the one just described is a *logical cut* (i.e. between two dual logical rules). The other forms of cut are when the rules preceding the cut are not dual. In this case, the Intuitionistic System eliminate the cut by lifting it in the premises of one of the rules (that becomes the last rule of the proof). These kind of cuts are "unobservable" in the IS's, namely they have no counterpart. So, let us concentrate on logical cuts only.

Let L and R be the left and right sequent in the cut-rule, respectively. The first observation is that, during the process of cut-elimination, the proofs ending into the

premises of L and R are considered as a whole: no assumption about them is done and every operation on any of the hypotheses (bound by L or R) must be done on the others in the same sequent, too. These sequents constitute the *interface* of the cut. Starting from the interface, one can imagine to build up a new proof, by means of arbitrary inference rules. In the case of implication we have used a sequence of two cuts. However, other choices could be possible.

The unique constraint of the cut-elimination process is the *prohibition of creating new hypotheses.* This has two implications:

1. the variables bound by L or R must be suitably filled in (typically with cuts or introducing *new* forms binding them);

2. if axioms are introduced then the variable in the premise must be consumed (with a cut or by another rule) by the proof.

What is the shape of the induced rewriting in the underlying IS? The lhs must be something of the form $\mathbf{d}(\mathbf{c}(\vec{x}^1_{k_1}.X_1,\cdots,\vec{x}^m_{k_m}.X_m),\cdots,\vec{x}^n_{k_n}.X_n)$ since a (logical) cut always involves a destructor rule and a constructor one. The X_i represent the proofs ending into the sequents in the hypotheses of L and R. In the right hand sides we may

- introduce new variables with axioms or with weakenings (denoted by x),

or, starting from proofs $H_1,\cdots,H_n$ that have been already built up,

- we can exploit proofs X_i (provided we fill the bound hypothesis: notation $X[^{H_1}/_{x_1},\cdots,^{H_n}/_{x_n}]$)

- or introduce a new logical rule (denoted as $\mathbf{f}(\vec{x}^1.H_1,\cdots,\vec{x}^n.H_n)$).

The other logical operations (contractions, cuts) are visible in the syntax under copying or interactions. The syntactical constraint reflecting the logical absence of new hypotheses is: right hand sides of rules must be closed expressions.

According to the above paradigm, every Intuitionistic System can be modeled through a suitable IS. Let us consider some example.

Example 3.8 (Naturals) Natural numbers are defined by two constructors 0 and succ. These constructors are respectively associated with the following right introduction rules:

$$(nat, right_0) \quad \vdash 0 : nat \qquad (nat, right_S) \quad \frac{\Delta, \vdash n : nat}{\Delta \vdash \mathsf{succ}(n) : nat}$$

In this case, we have two introduction rules for the type *nat*. A typical destructor is add.

$$(nat, left_{add}) \quad \frac{\Delta \vdash p : nat \qquad \Gamma, y : nat \vdash t : A}{\Delta, \Gamma\, x : nat \vdash t[^{\mathsf{add}(x,p)}/_y] : A}$$

where A can be any type. Of course, we have a different left introduction rule for each destructor. The following is an example of cut:

$$\dfrac{\vdash \mathtt{0} : nat \qquad \dfrac{\Delta \vdash p : nat \qquad y : nat, \Theta \vdash t : A}{\Delta, x : nat, \Theta \vdash t[^{\mathtt{add}(x,p)}/_y] : A}}{\Delta, \Theta \vdash t[^{\mathtt{add}(x,p)}/_y][^{\mathtt{0}}/_x] : A}$$

that is simplified into:

$$\dfrac{\Delta \vdash p : nat \qquad y : nat, \Theta \vdash t : A}{\Delta, \Theta \vdash t[^{p}/_y] : A}$$

The above elimination induces the IS-rule $\mathtt{add}(\mathtt{0}, X) \rightarrow X$, according to which we have that $t[^{\mathtt{add}(x,p)}/_y][^{\mathtt{0}}/_x]$ and $t[^{p}/_y]$ are equal. ■

Example 3.9 (Lists) Lists are defined by means of two constructors **cons** and **nil** of arity 00 and ε, respectively. The typical destructors are **hd** and **tl** of arity 0. In the case of lists of integers, we may write the following introduction rules for the type *natlist*:

$$(natlist, right_{nil}) \quad \vdash \mathtt{nil} : natlist$$

$$(natlist, right_{cons}) \quad \dfrac{\Delta \vdash n : nat \qquad \Gamma \vdash l : natlist}{\Delta, \Gamma \vdash \mathtt{cons}(n, l) : natlist}$$

$$(natlist, left_{hd}) \quad \dfrac{\Gamma, y : nat \vdash t : A}{\Gamma,\ x : natlist \vdash t[^{\mathtt{hd}(x)}/_y] : A}$$

$$(natlist, left_{tl}) \quad \dfrac{\Gamma, y : natlist \vdash t : A}{\Gamma,\ x : natlist \vdash t[^{\mathtt{tl}(x)}/_y] : A}$$

A typical cut is:

$$\dfrac{\dfrac{\Delta \vdash n : nat \qquad \Gamma \vdash l : natlist}{\Delta, \Gamma \vdash \mathtt{cons}(n, l) : natlist} \qquad \dfrac{\Theta, y : Nat \vdash t : A}{\Theta,\ x : natlist \vdash t[^{\mathtt{hd}(x)}/_y] : A}}{\Delta, \Gamma, \Theta \vdash t[^{\mathtt{hd}(x)}/_y][^{\mathtt{cons}(n,l)}/_x]}$$

and the obvious cut elimination rule gives:

$$\dfrac{\Delta \vdash n : nat \qquad \Theta, y : nat \vdash t : A}{\Delta, \Theta \vdash t[^{n}/_y] : A}$$

Again, by the reduction rule $\mathtt{hd}(\mathtt{cons}(X, Y)) \rightarrow X$, we have $t[^{\mathtt{hd}(l)}/_y][^{\mathtt{cons}(n,l)}/_x] = t[^{n}/_y]$. As an exercise the reader can provide the cut between $(natlist, right_{cons})$ and $(natlist, left_{tl})$ and verify that it induces the following rewriting:

$$\mathtt{tl}(\mathtt{cons}(X, L)) \rightarrow L$$ ■

3.3.2 ... and back

The *vice versa*, namely interpreting an IS into an Intuitionistic System is not always possible. In particular, the main problems are due to the lack of any type discipline in IS's (we have the same problem with the pure λ-calculus).

Up to this inadequacy, it is possible to provide the generic rule corresponding to a form. The paradigm is exactly the reverse of that discussed in the previous subsection. In particular, we respectively associate with a destructor or a constructor the two introduction rules described at the beginning of section 3.3.1.

A rewriting rule, is interpreted as the elimination of a logical cut. In order to understand the way the cut is rewritten in the intuitionistic system, we reason by induction on the structure of the rhs of the IS-rule. Recall that the right hand side H of an IS-rule is a closed expression built up by the following syntax:

$$H \quad ::= \quad x \quad | \quad \mathbf{f}(\vec{x}^1.H_1, \cdots, \vec{x}^n.H_n) \quad | \quad X[^{H_1}/_{x_1}, \cdots, ^{H_n}/_{x_n}]$$

The case of variables is easy: they correspond to axioms. A metaexpression $X[^{H_1}/_{x_1}, \cdots, ^{H_n}/_{x_n}]$ is interpreted as a sequence of cuts between the variables x_i in X and the proofs representing H_i. A metaexpression of the shape $\mathbf{f}(\vec{x}^1.H_1, \cdots, \vec{x}^n.H_n)$, where $\mathbf{f}$ is a constructor, is interpreted by taking the proofs corresponding to $H_1, \ldots, H_n$, possibly adding weakenings if bound variables do not appear in the bodies, and adding as last rule that corresponding to $\mathbf{f}$. If $\mathbf{f}$ is a destructor, the rule corresponding to $\mathbf{f}$ take as sub-proofs those of $H_2, \ldots, H_n$. Finally a cut must be introduced between the rule of $\mathbf{f}$ and the proof of H_1. Some cuts with axioms can be eliminated in the obvious way, after this rough interpretation.

The last step consists of adding a sequence of weakenings that perform the sharing of the copies of the proofs replacing the same metavariable.

4 The formal definition of IS

An Interaction System is defined by a *signature* Σ and a set of *rewriting rules* R.

(The signature) The signature Σ consists of a denumerable set of *variables* and a set of *forms*. The set of forms is partitioned into two disjoint sets Σ^+ and Σ^-, representing *constructors* (ranged over by $\mathbf{c}$) and *destructors* (ranged over by $\mathbf{d}$). Variables will be ranged over by x, y, z, $\cdots$, possibly indexed. Vectors of variables will be denoted by $\vec{x}_i$ where i is the length of the vector (often omitted).

Each form can work as a binder. This means that in the arity of the form we must specify not only the number of arguments, but also, for each argument, the number of variables it is supposed to bind. Thus, the *arity* of a form $\mathbf{f}$, is a finite (possibly empty) sequence of naturals (and not, as usual, a natural!). Moreover, we have the constraint that the arity of every destructor $\mathbf{d} \in \Sigma^-$ has a leading 0 (i.e., it cannot bind over its

first argument). The reason for this restriction is that, in Lafont's notation [15], at the first argument we find the *principal port* of the destructor, that is the (unique) port where we will have interaction.

Expressions, ranged over by $t, t_1, \cdots$, are inductively generated by the two rules below:

a. every variable is an expression;

b. if $\mathbf{f}$ is a form of arity $k_1 \cdots k_n$ and $t_1, \cdots, t_n$ are expressions then $\mathbf{f}(\vec{x}^1_{k_1}.t_1, \cdots, \vec{x}^n_{k_n}.t_n)$ is an expression.

Free and bound occurrences of variables are defined in the obvious way. As usual, we will identify terms up to renaming of bound variables (α-conversion).

(The rewriting rules) Rewriting rules are described by using schemas or *metaexpressions.* A metaexpression is an expression built up also with *metavariables*, ranged over by $X, Y, \cdots$, possibly indexed (see [1] for more details). Metaexpressions will be denoted by $H, H_1 \cdots$.

A *rewriting rule* is a pair of metaexpressions, written $H_1 \to H_2$, where H_1 (the *left hand side* of the rule, lhs for short) has the following format

$$\mathbf{d}(\mathbf{c}(\vec{x}^1_{k_1}.X_1, \cdots, \vec{x}^m_{k_m}.X_m), \cdots, \vec{x}^n_{k_n}.X_n)$$

and $i \neq j$ implies $X_i \neq X_j$ (*left linearity*). The arity of $\mathbf{d}$ is $0k_{m+1} \cdots k_n$ and that of $\mathbf{c}$ is $k_1 \cdots k_m$.

The *right hand side* H_2 (rhs, for short) is every *closed* metaexpression, whose metavariables are already in the lhs and built up by the following syntax

$$H \quad ::= \quad x \quad | \quad \mathbf{f}(\vec{x}^1_{a_1}.H_1, \cdots, \vec{x}^j_{a_j}.H_j) \quad | \quad X_i[{}^{H_1}/_{x^i_1}, \cdots, {}^{H_{k_i}}/_{x^i_{k_i}}]$$

The expression $X[{}^{H_1}/_{x_1}, \cdots, {}^{H_n}/_{x_n}]$ denotes a meta-operation of substitution, as in the λ-calculus.

Finally, the set of rewriting rules must be *non-ambiguous*, i.e. there exists at most one rewriting rule for every pair **d**-**c**.

Example 4.1 As we already remarked, the most typical example of IS is λ-calculus. Many interesting Interaction Systems can be then defined by enriching the λ-calculus with "δ-rules". For instance, an alternative way to look at the recursion operator μ is as a destructor of arity 0 interacting with λ (i.e., a destructor alternative to application). In this case it is described by the following reduction

$$\mu(\lambda(\langle x \rangle.X)) \to X[{}^{\mu(\lambda(\langle y \rangle.X[{}^{y}/_{x}]))}/_{x}]$$

(Note that the μ and the λ in the rhs have nothing to do with those in the lhs). We shall use this definition of μ in the rest of the paper. ■

4.1 Graphical representations

Expressions of Interaction Systems have graphical representations that are reminiscents of Lafont's Interaction Nets. In particular, a form $\mathbf{f}$ of arity $k_1 \dots k_n$ is represented as a node of name $\mathbf{f}$ with $1 + \sum_{i=1}^{n} p_i$ ports (edges); $p_i = k_i + 1$ is the i-th *partition* of $\mathbf{f}$. The i-th partition represents the connections between $\mathbf{f}$ and its i-th argument M. In particular, one connection is with the root of M, and k_i with the variables bound by $\mathbf{f}$ (the latters will be called *bound* ports of the partition). Observe that bound variables correspond to (bound) ports of the form $\mathbf{f}$. Thus our graphs are cyclic.

The unique port which does not belong to a partition is called the output port of $\mathbf{f}$. All the forms, have a *principal port*, which is drawn with an outgoing arrow (the arrow is omitted when it is clear from the context: see Figure 2 below. The other entries are called *auxiliary ports* (see [15]). In the case of a constructor, the principal port coincides with the output port. In the case of a destructor, the principal port is the unique edge in the first partition (recall that the arity of the first argument of a destructor is eventually 0, so this partition is a singleton and does not have bound ports).

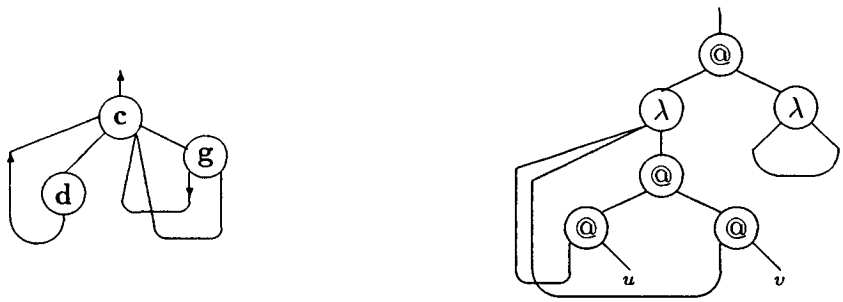
By this definition, interactions between constructors and destructors takes place only at principal ports (local sequentiality).

Following Lafont, it is possible to add *polarities* to ports. In particular, the output port of each form is always positive. So, the principal port of a constructor is positive. On the contrary, the principal port of a destructor is negative. All bounds port have positive polarities, and all the other ports are negative. In particular, in every partition we have *exactly* one negative port. An edge may only connects forms at ports with opposite polarities.

Polarities have a strong logical motivation. They are essentially related to the connections established by the form with the formulae (the conclusions) in the upper sequents of the associated introduction rule: positive if the formula (the conclusion) is in the rhs of a sequent, and negative otherwise). Moreover, ports belonging to a same partition are eventually connected with conclusions of a *same* sequent (see [3] for more details).

The correspondence between ports, bound variables and body of the arguments is fixed once and for all for each form. This means that all ports of a given form should be suitably "marked" (for the sake of readability, we shall generally omit to do that). For example the graphical representation of $\mathbf{c}(\langle x_1, x_2\rangle.\,\mathbf{d}(x_1), \langle y\rangle.\,\mathbf{g}(y, y))$ is illustrated in Figure 2.(a). Variables which are not bound will be depicted as dangling edges whose ends are labeled by the name of the variables. This is the case for the variables u and v in the λ-term $(\lambda x.(xu)(xv))(\lambda y.y)$ depicted in Figure 2.(b). We recall that, in this way, several edges may have a common end, due to the multiple occurrence of a free variable in an expression.

The reader is referred to [3] for more details about the graphical representation. A lot of examples will be found in the following pages.



(a) (b)

Figure 2: Graphical representations of expressions

5 Labeling and the family relation

This section is devoted to the generalization of Lévy's labeling [20] from λ-calculus to arbitrary Interaction Systems. Labeling allows us to define the family relation, that is the kind of "optimal" sharing the implementation should support.

Definition 5.1 Let $L = \{a, b, \cdots\}$ be a countable set of *atomic labels*. The set $\mathbf{L}$ of *labels*, ranged over by $\alpha, \beta, \cdots$ is defined by the following rules:

$$L \mid \alpha\beta \mid (\alpha)_s$$

where $s \in \mathrm{N}^+$. The operation of concatenation $\alpha\beta$ will be assumed *associative*. ■

Although its formalization is a bit entangled, the idea behind the following labeling is very simple. When a redex is fired, a label α is captured between the destructor and the constructor; this is the label associated with the redex. Then, the rhs of the rewriting rule must be suitably "marked" with α, in order to keep a trace of the history of the creation. Moreover, since in the rhs we may introduce new forms, we must guarantee a property similar to the initial labeling, where all labels are different. This means that all links in the rhs must be marked with a different function of α (and we shall use sequences of integers, for this purpose).

Let us come to the formal definition. Every IS (Σ, R) can be turned in a free way into a (*labeled*) CRS (Σ^L, R^L).

The forms of Σ^L are those in $\Sigma \cup \mathbf{L}$ with the arity of every $\alpha \in \mathbf{L}$ equal to 0. If

$$\mathbf{d}(\mathbf{c}(\vec{x}^1.X_1, \cdots, \vec{x}^m.X_m), \cdots, \vec{x}^n.X_n) \rightarrow H$$

is a rule in R then, for every i and for every i-tuple $\alpha_1, \cdots \alpha_i$, the rule

$$\mathbf{d}(\alpha_1(\cdots(\alpha_i(\mathbf{c}(\vec{x}^1.X_1, \cdots, \vec{x}^m.X_m)\cdots), \cdots, \vec{x}^n.X_n) \rightarrow \mathcal{L}^0_{\alpha_1\cdots\alpha_i}(H)$$

belongs to R^L, where $\mathcal{L}^s_\alpha$ is defined over metaexpressions as follows

$$\begin{aligned}
\mathcal{L}^s_\alpha(x) &= (\alpha)_s(x) \\
\mathcal{L}^s_\alpha(\mathbf{f}(\vec{x}^0.\, H_0, \cdots, \vec{x}^m.\, H_m)) &= (\alpha)_s(\mathbf{f}(\vec{x}^0.\, \mathcal{L}^{s0}_\alpha(H_0), \cdots, \vec{x}^m.\, \mathcal{L}^{sm}_\alpha(H_m)) \\
\mathcal{L}^s_\alpha(X[{}^{H_0}/_{x_0}, \cdots, {}^{H_n}/_{x_n}]) &= (\alpha)_s(X[{}^{\mathcal{L}^{s0}_\alpha(H_0)}/_{x_0}, \cdots, {}^{\mathcal{L}^{sn}_\alpha(H_n)}/_{x_n}])
\end{aligned}$$

Example 5.2 Consider again the λ-calculus. The β-reduction $@(\lambda(\langle x\rangle.\, X), Y) \to X[{}^{Y}/_{x}]$ gives rise, in the labeled version, to the following rules:

$$@(\alpha_1(\cdots(\alpha_i(\,\lambda(\langle x\rangle.\, X)\cdots), Y) \;\to\; \mathcal{L}^0_\ell(X[{}^{Y}/_{x}])$$

where $\ell = \alpha_1 \cdots \alpha_i$. Note that, by definition, $\mathcal{L}^0_\ell(X[{}^{Y}/_{x}]) = (\ell)_0(X[{}^{(\ell)_{00}(Y)}/_{x}])$, therefore, by replacing ℓ_0 with $\overline{\ell}$ and ℓ_{00} with $\underline{\ell}$, we easily recognize Lévy's labeling. ■

Labeled expressions are depicted by the same standard as unlabeled ones, with the agreement to write labels besides edges connecting forms.

Let (Σ, R) be an IS and let (Σ^L, R^L) be the labeled CRS built in the way described above. Given a form $\mathbf{f}$ in Σ and an occurrence of it in a term t of Σ^L, we say that $\mathbf{f}$ has *label* $\alpha_1\alpha_2\cdots\alpha_i$ if, in the syntactic tree of t, $\alpha_1\alpha_2\cdots\alpha_i$ is the path towards the root which links $\mathbf{f}$ to the less outside form in Σ (or to the root). The *degree* of a redex u is the label of the constructor (i.e. the sequence of the labels between the pair of symbols **d-c** of the redex u).

We will say that an expression owns the property **INIT** when the label of the forms are atomic and pairwise different.

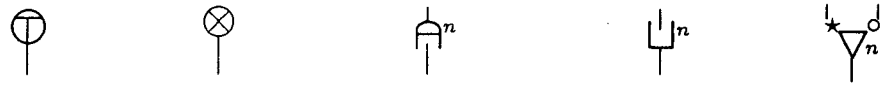
Definition 5.3 Two redexes yielded by a derivation starting at a labeled expression owning **INIT** are *in a same family* if and only if their degrees are the same. ■

This approach to the notion of redex-family based on labels does not give much insights about the intuitions that are behind. There are other equivalent approaches, suggested by the case of λ-calculus [20,21]. The relations among them have been discussed in detail in [3,19].

6 Sharing graphs

Let us come to the optimal implementation of IS's. As remarked in the Introduction, the aim is to share, along derivations, redexes that are in the same family. This is yielded by enriching the graphical representation of expressions with control operators. Such operators are described in Figure 3 and must be considered as forms. This means that each node has a principal port of interaction. In Figure 3, the principal ports are always at the lower edges.

To be formal, croissants brackets and fans are of two types, according to the polarity of their principal port. When the polarity of (the principal port of) the fan is negative



(root) (erasing) (croissant) (bracket) (fan)

Figure 3: The control operators

then the node is called *fan-in*; when the polarity is positive, the fan is named *fan-out*. Fans are the main nodes for implementing sharing.

You can get some intuition on control operators by their relation with linear logic. In this logic, every datum which has a not-linear use, must be put inside a *box*. The number of boxes enclosing a datum essentially expresses the number of different levels of sharing the datum is subject to.

The purpose of square brackets of index 0 is essentially that of marking some points of "discontinuity" in the graph which are not explicitly expressed by other control operators. Typically, when we pass from a variable to its binder, or from an application to its right argument (in both cases we are implicitly switching from a type $!(D)$ to D, or *vice versa*).

Boxes can be opened, or shifted inside other boxes. Both these operations dynamically modify the *sharing levels* in the term. So we must introduce some operators to implement these modifications. In particular, a box is opened when the datum it contains is accessed via a variable (one potential level of sharing has been dropped). This "push down" on the datum is expressed by the croissant. So, the translation of a variable will just look as follows:

A box M can be shifted inside another box N when we try to access M from some of the free variables of N. In this case, the levels of sharing M is subject to, is augmented of 1 (the potential sharing N). Again, we need a new operator to express this modification. This is the square bracket (with index $n \geq 0$). In particular, all the time we build a box around a datum P (every time a datum can be potentially shared), we must add a square bracket of index 1 on each negative conclusion (free variable) of P.

From the semantical point of view, brackets and croissants should be understood as *context transformers*. They modify the shape of the context (adding, erasing, freezing and unfreezing levels), in order to correctly travel along the sharing graph in the read-back phase. For example, the presence of indexes besides the operators indicates the *depth* where the modification takes place in the context (see Section 8 or [17,18]).

The rules governing the interactions between control operators are drawn in Figure 4.

The root and void nodes are respectively attached to the "important" and "unimportant" dangling edges of the graph. The important edges are the root and the free

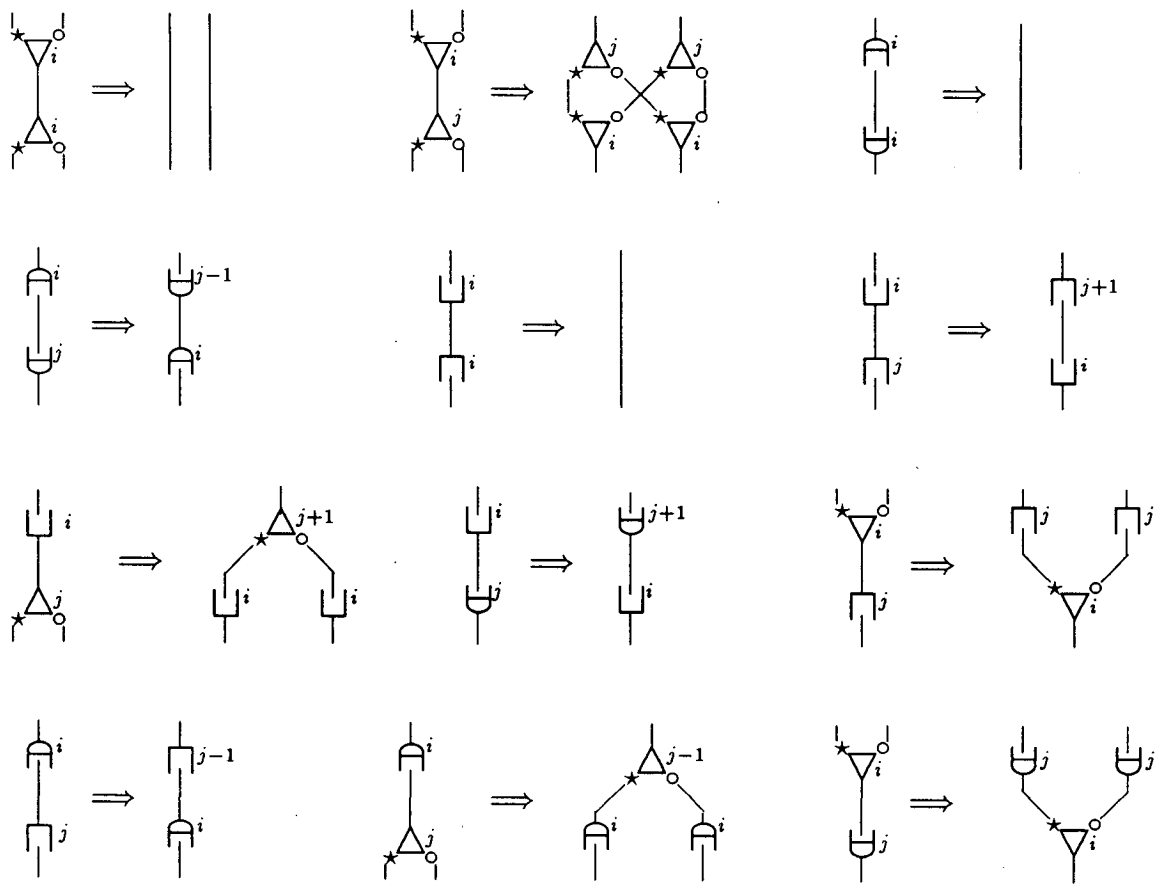


Figure 4: The control rules ($0 \leq i < j$)

variables; the useless edges are those parts of the graph that have been discarded along the derivation. Indeed, the main purpose of erasing is to connect part of the graph that are erased by a contraction in order to preserve locality of the rewriting rules. We could add rules providing garbage collection (mainly involving the erasing node), but these do not eliminate all garbage and are not essential for correctness. So we omit them.

7 Implementation

Now we have all the preliminaries to provide the implementation of a generic IS. The optimal implementation is described as a graph rewriting system. The nodes of the graph are either control operators or syntactical forms of the IS. Actually, the graph rewriting system is itself an Interaction Net, inheriting all good properties of this formalism (in particular, the strong diamond property).

The translation of IS-expressions is discussed in Subsection 7.1 below. Subsection 7.2

will deal with the rewriting rules.

7.1 The translation of expressions

In the translation of an arbitrary expression in sharing graph we shall essentially follow [11]. The translation function $\mathcal{T}^+$ calls an auxiliary function $\mathcal{T}$, inductively defined in Figure 6.

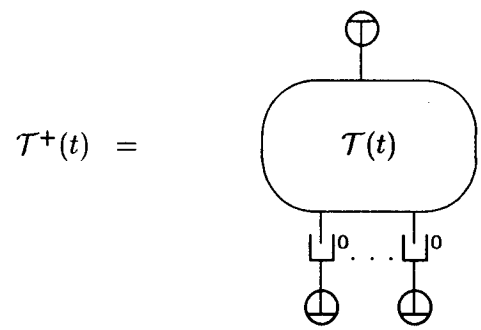


Figure 5: The encoding function $\mathcal{T}^+$

Remark 7.1 The definition of $\mathcal{T}$ will be slightly different from that provided in [4,19]. Actually, there, we followed strictly the local implementation of boxes for linear logic described in the first part of [11]. The reader can verify that translation eventually introduces a redundant number of redexes between brackets of index 0 facing each other. Therefore expressions in normal form were encoded, in general, by sharing graphs not in normal form. A better translation can be obtained by avoiding the introduction of all these redexes. Indeed, such translation relies on the implementation of Girard's *unified logic* (a synthesis of classical, intuitionistic and linear logic) described in the second part of [11]. This is the approach we will follow here. ■

For simplicity, in Figure 6, we consider the paradigmatic case when constructors and destructors have respectively arity 1 and 01. The other cases are easily derived. In this figure, the dangling edges in the bottom represent generic free variables which are not bound by the forms **c** and **d**. In particular, the edge outgoing the 1-indexed fan-in, represents a free variable which is common to M and N. If some bound variable does not occur in the body, the corresponding port of the binder is connected to an erasing node, as shown in Figure 7 (logically, the variable has been introduced by means of the weakening rule). The above translation is a more or less obvious consequence of the linear logic implementation in [11] (via a type isomorphism $D \cong (!D) \multimap D$. A variable x represents an axiom whose negative edge has been derelicted. All the arguments of forms (apart the argument at the main port of a destructor) must be put inside boxes (must be protected by !). This because, each one of these arguments may be used in a non linear way during rewriting, and/or can be used as an argument in a substitution.

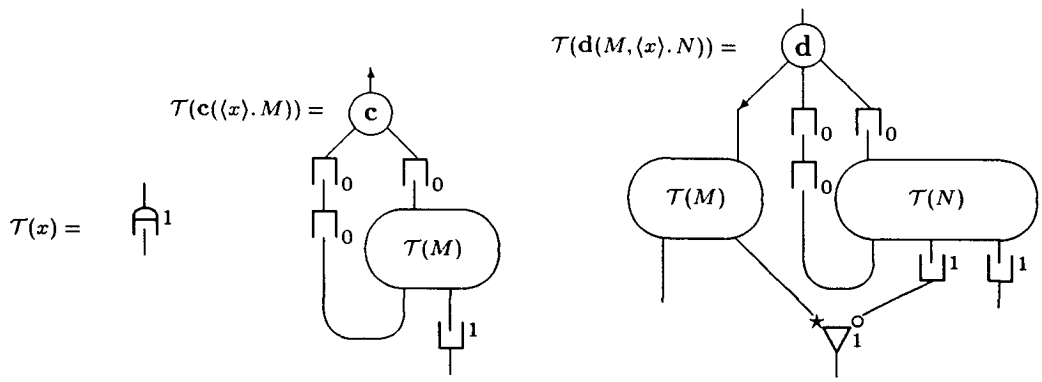


Figure 6: The translation function $\mathcal{T}$ (bound variables occur in the bodies)

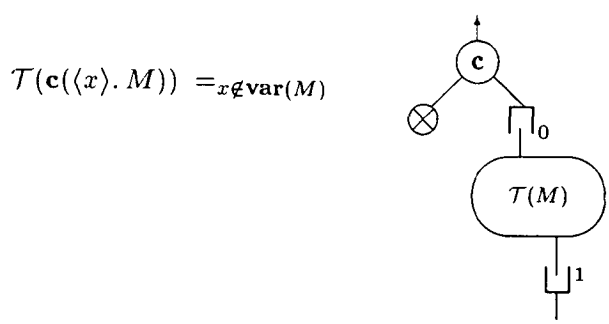
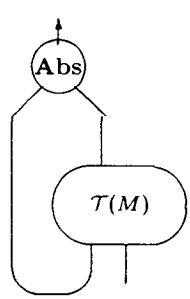


Figure 7: The translation function $\mathcal{T}$ (the bound variable does not occur in the body)

Here a peculiarity deserves to be emphasized. Let us consider the case of the constructor **c** (the same considerations hold for the destructor). When we put the argument of **c** inside a box, the control operators to be added at the level of the bound variable are different form the control operators to be added to free variables. The reader should intuititively imagine to have a pseudo-binder between the form and the body of the argument, as drawn below:



In this case, when we perform the !-introduction and the c-introduction, we obtain:

That is as the constructor had no bound variable! Since the pseudo-binder is a ghost, it disappears, the bracket traverses it and we obtain the translation of Figure 6.

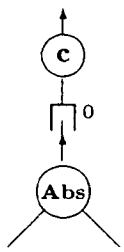


Figure 8: !-introduction with ghost-binder

The reason for proceeding in this way will become more clear when we will describe the translation of the rewriting rules of IS's. At that stage, the ghost-binder will become apparent and will play an essential role during partial evaluation.

7.2 The translation of rewriting rules

Rewriting rules may be classified in three groups. We shall discuss each group in a separate subsection.

7.2.1 Control Rules

These are the 12 rules in Figure 4. These rules provide the general framework for the optimal implementation of the structural part of IS's.

7.2.2 Interfacing Rules

These are the rules which describe the interaction between control operators and forms of the syntax (that is, they describe the interface between the structural and the logical part of IS's). These rules have a polimorphic nature. We define them by means of schemas, where **f** can be an arbitrary form of the syntax. The rules are drawn in Figure 9 (where $i > 0$).

As you see, interfacing the structural and the logical part of IS at the implementation level is *very* simple. This is a main consequence of the logical nature of IS's.

7.2.3 Proper Rules

These rules describe the interactions between destructors and constructors of the IS's. These are the only rules which are dependent from the particular Interaction System under investigation, and the only ones which deserve some care, in the translation. We shall define the implementation of the rewriting rules in four steps: β-expansion, linearization, translation and partial evaluation.

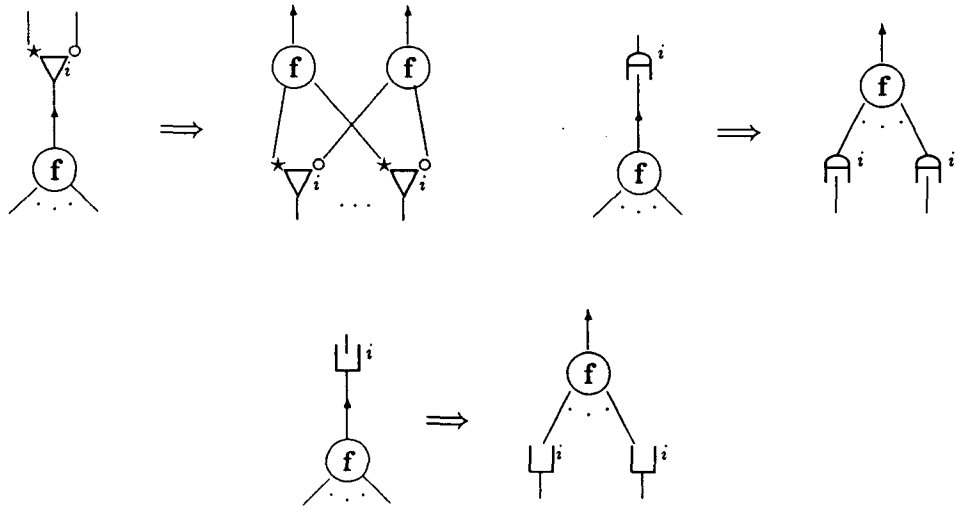


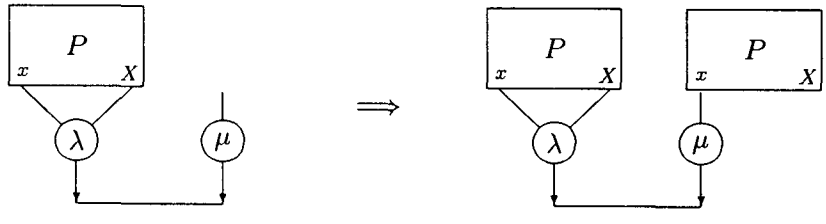
Figure 9: The interfacing rules between control operators and forms ($i > 0$)

The idea behind β-expansion and linearization is that of expliciting the "interface" between the new forms which have been possibly introduced in the rhs of the rule, and the metavariables in its lhs. Then, we may essentially translate the rhs as a normal term, just regarding the metavariables as "black box". Finally, we must partially evaluate the graph obtained in this way, since during β-expansion and linearization we have introduced some "pseudo-operators" which should disappear.

The linearization step is particularly important (β-expansion is just aimed to linearization). Consider the rewriting rule for μ:

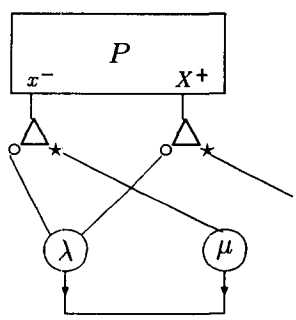
$$\mu(\lambda(\langle x \rangle . X)) \to X[^{\mu(\lambda(\langle y \rangle . X[^{y}/_{x}]))}/_{x}]$$

Note that, in the rhs, we have two occurrences of the metavariable X. Intuitively, one could expect to express the previous reduction by means of a graph rewriting rule of the following kind:



However, in this way, the portion of graph in the box (corresponding to the metavariable X) should be physically duplicated. Consequently, if we had a (actual or virtual) redex inside X, it is duplicated, too. Moreover, the rewriting rule would not be "local" anymore (it would not be in the Interaction Net form), since it requires a global operation on a box. So, we must try to share the double occurrence of X in the rhs. To this aim,

observe that also the variable x occurs twice (one for each instance). This means that we must be able to *unshare* the graph at this level since one occurrence has to be bound by the λ and the other has to be connected to the μ-operator. This double operation of sharing and unsharing is just the purpose of Lamping's *fan-in* and *fan-out* operators. Summing up, we expect to get an implementation of the rhs of the rule for μ that looks like:



Notice moreover that the rewriting rule has now a completely *local* behaviour: it merely modifies the connections of ports of the two forms yielding the redex.

The difficult problem, solved by the following translation procedure, is to introduce in the correct way the control operators (brackets and croissants) which ensure the right matching of fan-ins and fan-outs.

(**β-expansion**) The first step is to β-expand all substitutions in the rhs. The aim of this step is to provide a clean vision of all the metavariables in the rhs. For this purpose we shall use two classes of *pseudo-forms*: abstraction $\mathbf{Abs}_n$ and application $\mathbf{App}_n$, for $n \geq 0$. Pseudo-forms are similar to all other forms of the syntax. $\mathbf{Abs}_n$ is a constructor of arity n whilst $\mathbf{App}_n$ is a destructor of arity 0^{n+1}. As the reader could probably imagine, they generalize λ-calculus abstraction and application. Their interaction is expressed by the rule:

$$\mathbf{App}_n(\mathbf{Abs}_n(\langle x_1, \cdots, x_n \rangle . X), Y_1, \cdots, Y_n) \rightarrow X[{}^{Y_1}/_{x_1}, \cdots, {}^{Y_n}/_{x_n}]$$

In the following we shall always omit the index 1 in $\mathbf{Abs}_1$ and $\mathbf{App}_1$.

The step of β-expansion consists in rewriting the rhs of the IS-rule by β-expanding substitutions into interactions of the pseudo-operators $\mathbf{Abs}_n$ and $\mathbf{App}_n$.

Example 7.2 Consider the rewriting rule for μ:

$$\mu(\lambda(\langle x \rangle . X)) \rightarrow X[{}^{\mu(\lambda(\langle y \rangle . X[{}^{y}/_{x}]))}/_{x}]$$

The β-expansion of the rhs gives the following term:

$$\mathbf{App}(\mathbf{Abs}(\langle x \rangle . X), \mu(\lambda(\langle y \rangle . \mathbf{App}(\mathbf{Abs}(\langle x \rangle . X), y)))) \qquad \blacksquare$$

Note that, after the β-expansion, all metavariables are closed by pseudo binders, i.e. they become expressions of the following kind: $\mathbf{Abs}_n(\vec{x}.\, X)$.

(linearization) The next step consists in *linearizing* the rhs w.r.t. the occurrences of expressions $\mathbf{Abs}_n(\vec{x}.\, X)$. This is obtained by taking, for every metavariable X_i occurring in the left hand side of the IS-rewriting rule (let them be k), a fresh *pseudo-variable* w_i and replacing every occurrence of $\mathbf{Abs}_n(\vec{x}^i.\, X_i)$ with w_i. In this way we yield a metaexpression T. Next T is closed w.r.t. the metavariables w_i's, and the (closed) metavariables $\mathbf{Abs}_n(\vec{x}^i.\, X_i)$ are passed as arguments to this term. In other words, by linearization, we get a metaexpression of the following kind, where each metavariable occur exactly once (and no substitution is applied to them):

$$\mathbf{App}_k(\mathbf{Abs}_k(\langle w_1, \cdots, w_k\rangle.\, T),\, \mathbf{Abs}_{n_1}(\vec{x}^1.\, X_1), \cdots,\, \mathbf{Abs}_{n_k}(\vec{x}^k.\, X_k))$$

where n_i is the arity of the metavariable X_i.

Example 7.3 After the linearization step, the rhs of the recursion rule becomes:

$$\mathbf{App}(\mathbf{Abs}(\langle w\rangle.\, \mathbf{App}(w,\, \mu(\lambda(\langle y\rangle.\, \mathbf{App}(w,\, y))))),\, \mathbf{Abs}(\langle x\rangle.\, X)) \qquad \blacksquare$$

We want to remark that every metavariable in the lhs of the IS-rewriting rule occurs exactly once in the linearized metaexpression yielded by the above procedure, even if it does not occur in the rhs of the IS-rewriting rule. For instance, in the case of conditionals, the linearization of the rhs of $\natural(\mathbf{T},\, X,\, Y) \to X$ gives

$$\mathbf{App}_2(\mathbf{Abs}_2(\langle w_1,\, w_2\rangle.\, w_1),\, \mathbf{Abs}_0(X),\, \mathbf{Abs}_0(Y)).$$

The actual erasing will be performed in the following steps (see translation and partial evaluation).

(translation) This step provides the graphical representation of the rhs of the rule. It is essential that, during the translation, we may consider each subexpression $\mathbf{Abs}_n(\vec{x}.\, X)$ as a "black-box". According to the linearization step, the expression that results will have the shape $\mathbf{App}_k(M,\ \mathbf{Abs}_{n_1}(\vec{x}^1.\, X_{i_1}), \cdots,\, \mathbf{Abs}_{n_k}(\vec{x}^k.\, X_{i_k}))$. The translation of this expression is drawn in Figure 10, where, for simplicity, we have assumed $n_i = 1$, for every i. Notice that metavariables are not put inside boxes: they *are* boxes, due to the translation of expressions in Figure 6. We implicitly use this box instead of building a new box around the argument of the application. In particular, no operation around the (unaccessible!) free variables of the instance of the metavariable must be performed.

Now we can provide some more intuition about our translation in Figure 6, and the role of the "ghost-binder". In particular, ghost-binders become apparent in the translation in Figure 10: they are the **Abs** pseudo-forms. The square bracket around each **Abs** are meant to extend the box containing the metavariable up to comprising the pseudo-abstraction, according to Figure 8.

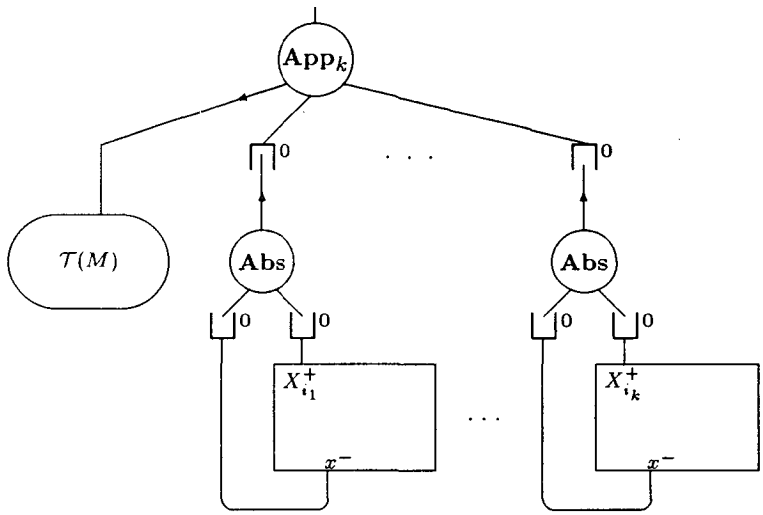


Figure 10: The translation step

The reason for introducing ghost-binder when translating rules, instead of when translating terms, is that we may now partially evaluate the rhs, eliminating all pseudo-forms which have been just introduced for convenience. This is the purpose of the next, final phase.

However, before describing partial evaluation, we must generalize the translation function $\mathcal{T}$ to pseudo abstractions and pseudo applications. The translation follows the usual implementation of the λ-calculus (since the body of a pseudo-abstraction is used linearly in β-reduction, the translation can be slightly simplified w.r.t. the general translation of "proper" IS-forms). This is described in Figure 11.

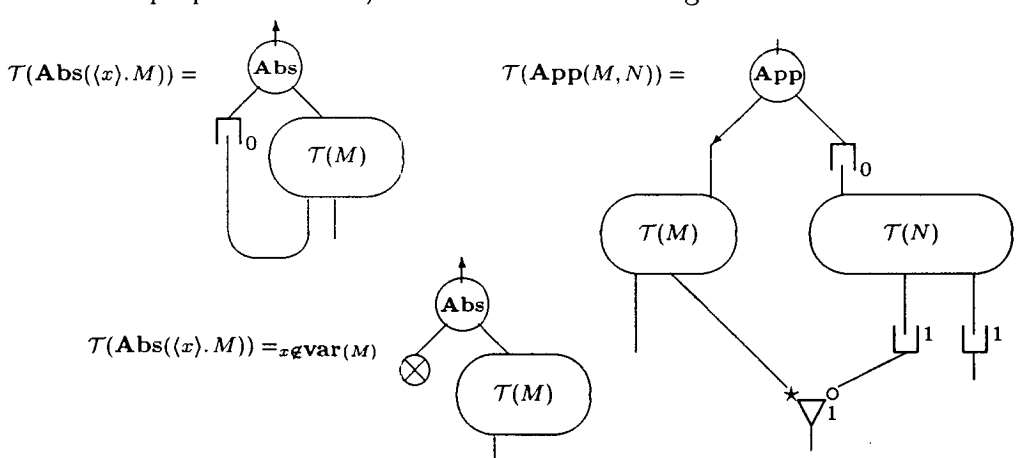


Figure 11: The generalization of the translation function $\mathcal{T}$

Now we can pursue on our running example, namely the implementation of the right hand side of the rule corresponding to redexes μ-λ. In Figure 12 we have depicted the sub-graph corresponding to the first argument of the outermost **App**, i.e.

$\mathcal{T}(\mathbf{Abs}(\langle w\rangle.\,\mathbf{App}(w,\,\mu(\lambda(\langle y\rangle.\,\mathbf{App}(w,\,y))))))$.

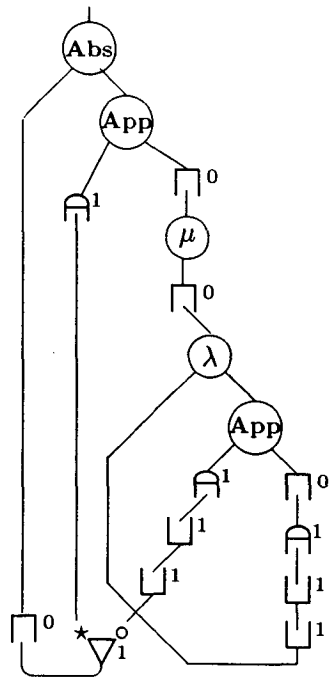
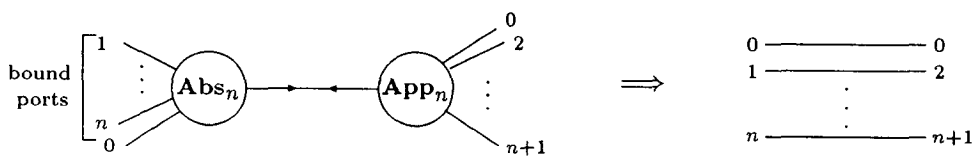


Figure 12: A part of the translation step of the rhs of μ-λ

A final observation before discussing partial evaluation. As already said, some metavariables in the lhs of the IS-rewriting rule could not occur in the rhs (e.g. the case of conditionals). According to the translation of **Abs**, the corresponding pseudo-variable introduced in the linearization step is implemented by an erasing node (since it does not occur in the body of the leftmost outermost **Abs**. This implies that, during the next phase, the corresponding expression is erased by the rule.

(**partial evaluation**) The final step is to partially evaluate the term we have obtained after the translation w.r.t. all pseudo operators. Recall that the reduction rule for pseudo application and abstraction is



Note that n can be 0. Here the rewriting rule simply consists in connecting the two edges coming into the auxiliary ports of $\mathbf{Abs}_0$ and $\mathbf{App}_0$.

Proposition 7.4 The partial evaluation of the expressions yielded by the translation step strongly normalizes to a graph without pseudo-forms.

Proof: After the linearization step, we yield an expression

$$\mathbf{App}_k(\mathbf{Abs}_k(\langle w_{i_1}, \cdots, w_{i_k}\rangle.\, T),\, \mathbf{Abs}_{n_1}(\vec{x}^{\,1}.\, X_{i_1}), \cdots,\, \mathbf{Abs}_{n_k}(\vec{x}^{\,k}.\, X_{i_k}))$$

where the expression T exploits only pseudo-forms **App**. Moreover, every occurrence of these pseudo-forms, have the shape $\mathbf{App}_k(w_X,\, M)$. By firing the unique **App-Abs** pair in the graph yielded by the translation step, every occurrence of $\mathbf{App}_k(w_X,\, M)$ becomes "almost" a redex. That is, there is a sequence of 1-indexed fan-ins and 1-indexed croissants (0-indexed brackets can be eliminated by means of the control rules) along the path connecting the principal ports of **App** and **Abs**. These control operators can be pushed inside the **Abs** pseudo-form by means of the interfacing rules. In this way we can fire every pair **App-Abs** thus yielding a normal form w.r.t. the partial evaluation. ■

The above proposition is a more or less obvious consequence of the fact that all pseudo-operators have been created by β-expansions (and the correctness of the translation, of course).

Example 7.5 By applying the previous technique (and some optimizations not worth discussing here) we obtain the implementation of the rhs of the rule concerning μ illustrated in Figure 13. ■

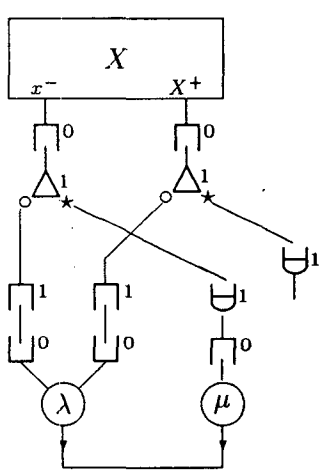


Figure 13: The graphical representation of the rhs of the rule firing μ-λ

Remark 7.6 The previous translation could (and must) be improved. Apart from studying optimization techniques for reducing the number of sharing operators, the translation should be *relativized* to the particular IS's under investigation. In particular, some operators of the syntax could make only a linear use of some of their arguments. For instance, this is the case of the λ-calculus, where the body of the abstraction is

treated linearly in β-reduction. These linear arguments have a simpler translation, since there is no need to put them inside a "box". However, in order to conclude that some operator **f** behaves linearly over one of its arguments we must examine *all* the interaction rules involving **f**. For instance, if we extend the λ-calculus with the μ operator, considering it as a destructor for λ as in the example above, the body of each λ should be put inside a box, since it can be duplicated when the λ interacts with μ. This is not the case with the other implementations of μ discussed in section 3. Thus, The choice of the rewriting system may have a big impact on the practical efficiency of the implementation. ■

8 Correctness

Let G be a sharing graph "representing" an IS-expression t. The implementation described in the previous section is *correct* if a graph-rewriting $G \to G'$ simulates a (possibly empty) set of IS-rewritings $t \twoheadrightarrow t'$ such that t' is the expression "represented" by G'.

It is clear that denoting the IS-expression represented by a sharing graph is an essential prerequisite for stating correctness. This is the so called *read-back* problem. It is solved in [17,10,11] by labeling edges of the sharing graphs through *contexts* and interpreting control operators (and forms, in [10,11]) as *contexts transformers*. Expressions matching the sharing graphs are thus "unfoldings" of the graphs where only *consistent paths* are considered, that is paths that behave well w.r.t. contexts.

8.1 Context semantics, access paths and read-back

Definition 8.1 The set of *contexts* over a set $\mathbf{X}$ of variables is inductively generated by the following rules:

- $\Box$ is a context (the *empty* context);
- if a is a context then so are $\circ \cdot a$ and $\star \cdot a$;
- if a and b are contexts then also $\langle a, b\rangle$ is a context;
- every variable $\mathbf{x} \in \mathbf{X}$ is a context. ■

Contexts will have the shape $A = \langle \cdots \langle a_n, a_{n-1}\rangle \cdots, a_0\rangle$ and we will say that a_n is the subcontext of A at *width* n (notation $A^n[a_n]$). Contexts are the data modified by control operators when traversed. As we will see in the following Definition 8.2, control nodes can be easily understood as context transformers. In particular, the traversal of a control node A can be forbidden if the external context does not allow the transformation performed by A. As a consequence, there are illegal paths in the sharing graph. Exploiting this idea, Lamping provides his read-back procedure (see [18]).

Since in our translation we used Gonthier's simplified set of control operators (and rewriting rules), the natural idea was to generalize the proof in [10] from λ-calculus to IS's. The notion of consistent path in [10] is much more informative and more complex than Lamping's one since also the forms of the syntax (application and abstraction), are regarded as context transformers (actually, for the particular shape of β-reduction, they can be safely assimilated to fans). In particular, a consistent path between an application and a λ corresponds to a *virtual redex* [22,5,7] (a virtual redex of a term t is a redex that does not exist yet in t, but that could be created along some derivation from t; the relation between consistent paths and virtual redexes has never been formally proved, but there seems to be good evidence for it). The interesting invariant w.r.t. reductions is that *the consistency of a path is not changed by firing control rules in Figure 4* (or β-reductions) [10,11]. This invariance provides a first rudimentary semantics, named *context semantics*, that gives the soundness of the graph reductions w.r.t. contexts (roughly, since we preserve virtual redexes until they are fired, the implementation respects the intended "operational behaviour" of the term).

Context semantics is still too weak w.r.t. correctness. However it is possible to use it in a giudicious way (see [10]). In particular, in [10], the authors take *Bohm-trees*, a standard semantics of λ-calculus that is invariant w.r.t. reductions and that gives tree-representations of λ-expressions. Then, they prove that the Bohm-tree representing a λ-term can be read-back from the sharing graph by taking *via via* consistent paths that end into the bound port of an unmatched abstraction. Due to the context semantics, such paths can be found directly in the initial graph, by starting at the opportune node.

Unfortunately, due to the generality of IS-rewriting rules, it does not seem to be possible to consider any longer forms of the syntax as context transformers (surely they cannot be assimilated to fans). This because IS-contractions may introduce new forms and new edges (therefore new paths). As a consequence, IS-virtual redexes cannot be described as *connected paths* in the original term of the derivation, as it is the case in the λ-calculus (see [5]).

For this reason, our approach will be closer to Lamping's original one. However, we use a set of control operators that is strictly contained in those used by Lamping (so, both the implementation and the proof are quite different). On the other side, the most manifest difference w.r.t. [10] is our read-back procedure: it provides real terms rather than Bohm-trees.

In conclusion, our proof not only generalizes the current approach to a much wider class of rewriting systems, but also, in our opinion, sheds some more light on correctness in the particular case of pure λ-calculus (putting in evidence some "magical" properties of this calculus).

Definition 8.2 (access path) An *access path* in a sharing graph G is a *directed* path, starting and ending respectively at a negative and positive port of proper forms and such that every edge of the path is labeled with a context and consecutive pairs of edges

satisfy one of the following constraints:

1. $A^i[b] \xrightarrow{\ i\ } A^i[\langle b, \Box\rangle]$

2. $A^i[\langle\langle b, a\rangle, c\rangle] \xrightarrow{\ i\ } A^i[\langle b, \langle a, c\rangle\rangle]$

3. $A^i[\langle b, a\rangle] \longrightarrow \star \cdot \xrightarrow{\ i\ } A^i[\langle b, \star \cdot a\rangle]$

4. $A^i[\langle b, a\rangle] \longrightarrow \circ \cdot \xrightarrow{\ i\ } A^i[\langle b, \circ \cdot a\rangle]$

5. $A \longrightarrow (\mathbf{f}) \longrightarrow A$ (provided that the path enters the auxiliary, not bound port)

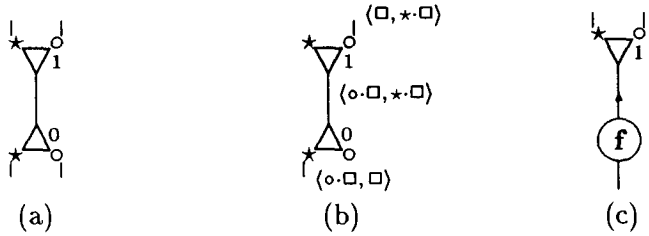
6. if the path enters the principal port of a form with context $\langle A, \langle B, C\rangle\rangle$ then it outgoes from an auxiliary negative port of **f** with context $\langle A, \langle B, \mathbf{x}\rangle\rangle$, where $\mathbf{x}$ is a fresh variable.

Access paths will be taken equivalent up to contexts. That is, two access paths having pairwise equal edges are considered equal, even if the contexts differ. ■

Note that, it is not possible to traverse a form **f** through one of its bound ports: when a path arrives in front of a bound port, it stops there. Actually the path should continue into the expression that the reduction of **f** substitute for the bound variable, but we need the evaluation of the term in order to determinate, in general, this expression. For this reason we preferred the above solution. The situation is better when we must access to the argument of **f**. In the following we will show that the *meaningful* part of the context at the principal port of **f**, when it is fired, is the same of that marking the output edge of the argument of **f**, if it will be called. So, in item 6, we are able to determinate the context at the auxiliary negative port of **f**, provided we know the context at the principal edge (that coincides with the context at the output of **f**, by item 5).

We remark that access paths are direct. This because item 6 cannot be defined for undirect ones.

The above definition put in evidence the important role of context transformers played by the control operators. Let us see with some examples how the contexts, modulo the control nodes, guarantee the proper matching between fans. Consider the sharing graph in Figure 14.(a). In Figure 14.(b) we have labeled the $\circ$-branch of the 1-indexed fan, the edge connecting the two fans and the $\star$-branch of the 0-indexed fan such that the corresponding path is an access path. That is the two fans do not match.



(a) (b) (c)

Figure 14: Sharing graphs

The sharing graph in Figure 14.(c) has a node that is a proper form **f** with one auxiliary edge. By definition, there is an access path that traverses **f** "from the top" (i.e. from the principal port: in this case, **f** must be a constructor and the auxiliary port is negative). Provided that the auxiliary edge is positive, there are two access paths that that traverse **f** "from the bottom" to the top: one outgoing the $\star$-branch of the fan and the other outgoing the o-branch (notice that **f** must be a destructor in this case). We leave to the reader the charge of finding contexts.

8.1.1 The read-back

A sharing graph is read-back by taking access paths. The idea is to start with a context that is as much unspecified as possible and instantiating it as the path grows. The algorithm below is nondeterministic, in the sense that whenever there are several possibilities for growing the path up, each one of them is pursued.

Definition 8.3 (The read-back) The *read-back* $\mathcal{R}$ from a sharing graph N to a (graphical representation of an) expression E is a procedure consisting of the following steps:

(1) let φ be the access path beginning at the root of N, yielded by specifying the initial context $\langle \mathbf{x}, \langle \mathbf{y}, \mathbf{z} \rangle\rangle$ and never traversing proper forms. Let **f** and **g** be the initial and ending forms of φ. There exist two forms $\mathbf{f}'$ and $\mathbf{g}'$ of the same type in E such that $\mathcal{R}(\mathbf{f}) = \mathbf{f}'$ and $\mathcal{R}(\mathbf{g}) = \mathbf{g}'$ and a unique edge $\mathcal{R}(\varphi)$ connecting $\mathbf{f}'$ and $\mathbf{g}'$ through the same ports and not traversing any other form;

(2) reiterate (1) starting from the negative ports of every form **g** with a context fitting with the constraints of Definition 8.2.

The procedure $\mathcal{R}$ induces a *read-back function* $\phi_{\mathcal{R}} : \mathcal{R}(N) \to N$ that maps every $\mathcal{R}(\mathbf{f})$ into **f** and every edge $\mathcal{R}(e)$ into e. ■

Observe that the initial context is built with variables. This allows us to specialize it as we fall inside arguments of forms (which are surrounded by brackets, that means requiring a further level of context). For instance, take the λ-expression $@(x, @(x, x))$ and its sharing graph representation, according to the mapping $\mathcal{T}^+$. Then, starting at

the root, we can outgo from the second argument of the outermost @. Here we meet a closed 0-indexed bracket. This requires that the 0-level context has the shape $\langle A, B\rangle$. Actually this is the case. But after traversing the square bracket, the context becomes $\langle\langle \mathbf{x}, \mathbf{y}\rangle, \mathbf{z}\rangle$. Now, in order to preserve the property that contexts have always the shape $\langle A, \langle B, C\rangle\rangle$, we must specialize $\mathbf{z}$ into $\langle \mathbf{z}_1, \mathbf{z}_2\rangle$ (intuitively, from the linear logic point of view, we are entering into a new box, i.e. a new area of memory). This operation should be not possible if we started with a context like $\langle \square, \langle \square, \square\rangle\rangle$, for example.

The first, easy property of the previous read-back is its consistency w.r.t. our translation $\mathcal{T}^+$. This is a boring check by structural induction on expressions, so we simply state the result, omitting the proof.

Proposition 8.4 Let t be an expression and E be its graphical representation. Then read-back $\mathcal{R}(\mathcal{T}^+(t)) = E$. ∎

8.2 Dynamic correctness of the read-back

The first "dynamic" result is a sort of *weak* context semantics: the invariance of the read-back (or of access paths) w.r.t. control and interfacing rules.

Proposition 8.5 Access paths (and hence the read-back procedure) are invariant w.r.t. control rules and interfacing rules.

Proof: It easy to verify that every access path traversing a redex in Figure 4 or Figure 9 is such that the contexts before the redex and after the redex do not change along the contraction. ∎

Note that the above property is false when proper redexes are considered. Take for instance λ-calculus and the expression $@(\lambda(\langle x\rangle.\,\mathbf{I}), M)$. Then there is an access path from the root to the argument M. However, after the reduction, there is no access path between the root and M, since M is disconnected. We warn the reader that, in the following, despite of this limitation, we shall still call *context semantics* (forgetting the prefix "weak") the invariant of the above proposition.

Correctness w.r.t. proper rules is much more difficult. Before addressing this problem, let us consider an example.

Take a graph with a redex μ-λ. Firing this redex results in replacing the subgraph determinated by the redex with the instance of the rhs's of the rule contracting μ-λ in Figure 13. Let G' be the ending sharing graph. The two 1-indexed fans generated by the rewriting should be paired by reading-back G' (because they are generated by duplicating the same fan along the partial evaluation of the reduction).

Back to our implementation of rhs's, it can be proved (see the proof of Lemma 8.9) that sharings, in every graph reachable from $\mathcal{T}^+(t)$, are always positively-indexed fans. So, a sufficient condition for the proper matching of fans is "what is at level greater than 0 is not modified by traveling inside an expression represented by a metavariable

and the two 0-indexed brackets that surround it". Lemma 8.8(1) will show that this property holds.

There is another subtle problem. Before firing the above redex μ-λ no access path traversing the bound port of the λ-node does exists. After the reduction that port is no more bound and every path ending there must be lengthened in order to fulfill the constraints of Definition 8.2 (i.e. it must end into a positive port of a proper form). In Lemma 8.8(2) we show (roughly) that the information at the 0-level of the initial context of an access path is not meaningful for its definition. So we can *join* two access paths, provided that they have the same meaningful contexts. For instance, in the case of the redex μ-λ, an access path ending into the bound port of λ can be joined with another starting at the form μ.

Two preliminary definition are needed. We have to define what is the meaningful context and the notion of access path from a binder of a variable x to some instance of x.

Definition 8.6 Let a be a context whose shape is $\langle A, \langle B, C\rangle\rangle$. The *calling context* of a is the context $\langle A, B\rangle$. C will be named the *offset context.* ■

Definition 8.7 A *loop* is an access path starting at a negative port of a form $\mathbf{f}$ and ending at a positive port in the same partition. ■

Lemma 8.8 Let N be a graph yielded along a derivation starting at $\mathcal{T}^+(t)$.

(1) For every loop φ, the initial and final calling contexts are equal. In particular, the initial context has always the shape $\langle \mathbf{x}_0, \langle \mathbf{x}_1, \cdots \langle \mathbf{x}_n, \mathbf{x}_{n+1}\rangle \cdots\rangle\rangle$ and the final context has the shape $\langle \mathbf{x}_0, \langle \mathbf{x}_1, \langle A, \mathbf{x}_{n+1}\rangle\rangle\rangle$, for some context A. Every edge of φ is labeled by a context whose shape is $\langle C, \langle B_1, \cdots, \langle B_m, \mathbf{x}\rangle \cdots\rangle\rangle$ and every context B_i is inessential, in the sense that, if we replace them with other contexts, we still obtain the same path.

(2) Every access path starting at the auxiliary negative port of a form does not depend from the offset context whose shape is always $\langle B_1, \cdots \langle B_m, \mathbf{x}\rangle \cdots\rangle$ $(m \geq 0)$. That is, replacing B_i with other contexts, we still obtain the same path.

Proof: By induction on the length of the derivation.
(**basic case**) Both (1) and (2) can be easily proved by structural induction over $\mathcal{T}$.
(**inductive case**) (1) The only interesting case is when we contract proper redexes because, by Proposition 8.5 both control rules and interfacing rules do not modify access paths.

Let us start with arguing above the rhs of the rewriting rule where the partial evaluation has not yet been performed (hence we have also pseudo-forms). Let us call N'_p the graph N where the proper redex (firing in $N \longrightarrow N'$) is replaced by the rhs of the rule that has not been partially evaluated.

Note that the new loops over pseudo-forms eventually satisfy the proposition, since either they are inside a portion of graph which is defined by means of $\mathcal{T}$ (thus we fall in the basic case), or they are internal to metavariables, and we use the inductive hypothesis (we have just replaced a binder with a pseudo-binder).

The only problematic case is that of loops over a binder external to the redex, ending into a variable internal to the instance of a metavariable, that is, the case of free variables that are internal to expressions as $\mathbf{Abs}_m(\vec{x}.X)$ (note that no other loops can be created by firing a proper rule). Let x be a free variable in $\mathbf{Abs}_m(\vec{x}.X)$. If we have a loop over this variable, it can be split in three parts: the access path φ_1 from the external binder $\mathbf{f}$ to the outermost **App**, the access path φ_2 from **App** to the metavariable X, the access path φ_3 internal to the instance of the metavariable. φ_2 traverses in order an open and a closed 0-indexed brackets. Therefore they have no effect over the initial context of φ except for requiring that its shape must be $\langle\bullet, \langle\bullet, \bullet\rangle\rangle$, that is guaranteed by hypothesis.

Let us now consider the partial evaluation. To this aim, reduce the unique redex between pseudo-forms of N'_p. After the reduction, the problem is again the presence of a free variable x in X bound by an external form $\mathbf{f}$. Let us consider the same example as above. Let then φ_1 be the access path from $\mathbf{f}$ to the outermost pseudo-application. After the reduction, the new loop on $\mathbf{f}$ must traverse a loop ψ of the pseudo lambda interacting with the outermost application, and then prosecute with the access path φ_2 inside $\mathbf{Abs}_m(\vec{x}.X)$, up to the variable x. Notice that φ_1 and ψ can be safely joined since $\varphi_1\psi$ is an access path at the beginning of the partial evaluation (easy check). Let us concentrate on the joining of ψ and φ_2.

By induction hypothesis (1), ψ does not modify the calling context. This means that the calling context at the beginning of φ_2 is the same as at the end of φ_1 and of ψ. Let $\langle C, \langle B_0, \cdots, \langle B_k, \mathbf{x}\rangle\cdots\rangle$, for some k, be the ending context of ψ. Recall that the initial context of φ_2 is $\langle C, \langle B_0, \langle \mathbf{x}_1 \cdots \langle \mathbf{x}_r, \mathbf{x}_{r+1}\rangle\cdots\rangle\rangle\rangle$ by inductive hypothesis. Thus take the $\mathbf{max}\{k, r\}$. Let it be k, for instance. Then we can "specialize" $\mathbf{x}_i$, $1 \leq i \leq r+1$, without altering the path (by hypothesis). That is $\mathbf{x}_j = B_j$ $(1 \leq j \leq r)$ and $\mathbf{x}_{r+1} = \langle B_{r+1}, \cdots, \langle B_k, \mathbf{x}\rangle\cdots\rangle$. Evidently, at the end of the grafting, we yield a consistent context $\langle C, \langle B_1, \langle A, \mathbf{x}\rangle\rangle\rangle$. By induction (2), starting with this context (possibly by specializing $\mathbf{x}$), it is possible to label consistently the tail φ_3 of ψ (i.e. the path connecting φ_2 with the bound port of $\mathbf{f}$).

In the rest of the paper we shall call *grafting* this operation of insertion of a loop inside an access path.

The case when $r = \mathbf{max}\{k, r\}$ is similar. Also the case when the other reductions **App-Abs** of the partial evaluation are performed can be proved in a similar way.

(2) Again, the only interesting case is when the last rule in the derivation is a proper rule (interfacing rules preserve the property since control operators have always indexes greater than 0). The reduction of **App-Abs** pairs may create new access paths by

"grafting" some loop ψ over **Abs** inside an old access path traversing **App** and going into some of its arguments. Consider such an access path φ starting at the auxiliary negative port of some form. It can be decomposed in φ_1 (the path leading to **App**) and φ_2 (the path prosecuting inside some argument of **App**). Note now that if $\varphi_1\psi$ is an access path after the reduction, it also was before, since

i. the final calling context of φ_1 coincide with the initial calling context of ψ and

ii. the offsets can be unified (in the same way as done previously in (1)).

Similar reasoning for the joining of ψ and φ_2. Notice that, as a particular case of this proof, it is possible to take φ_1 and φ_2 empty. Then the access path ϕ is joined with the output of a metavariable X by the pseudo-reduction. Obviously it remains a (part of an) access path provided there exists an access path starting at the output of the expression represented by X. ■

Lemma 8.8 is the counterpart of the *transparency property* of Lamping [18]. In particular, Lamping had a special operator (a global bracket), for dropping the offset near the variable-end of a loop. Pursuing this idea, with simple syntactical modifications, we obtain a stronger theorem, stating that any loop does not modify the *whole* context. However, this operator is not relevant during the computation; on the contrary, it introduces some annoying problems, since it must be properly "erased" every time we open the loop to perform a grafting.

(Correctness) The statement of correctness is based on two lemmas. The first states the absence of *deadlock situations* due to presence of two not interacting agents. The second ensures that rewriting a proper redex in the sharing graph N yields a graph that is read-back into the rewritten expression $\mathcal{R}(N)$.

Lemma 8.9 Let N be a sharing graph yielded through a derivation starting at $\mathcal{T}^+(t)$, for some t. Let e be an edge connecting a destructor/constructor (also **App**/**Abs** forms) pair and having a sequence of control operators in the way. If the edge is read-back into a redex then the control operators along e can be removed.

Proof: We must prove two properties: firstly that, by using control and interfacing rules, we do not yield a situation in which the principal port of a 0-indexed control operator faces the principal port of the constructor or destructor. Then that different control operators, labeled with the same index, never face each other along a redex.

By induction assume that, for every form in N, the principal edge could be rid of control operators along it whose principal edges have opposite directions. Moreover, assume that every auxiliary negative port of a destructor and a constructor faces (or could face, by performing control rules) *exactly* an open 0-indexed bracket (except the pseudo-form **Abs**) and every bound edge faces (or could face) the bound port with *exactly* two open 0-indexed brackets (except **Abs** that has one 0-indexed bracket). It is easy to verify that $\mathcal{T}^+(t)$ satisfies these properties.

Let us verify that redexes *created* by the (proper) rule $N \longrightarrow N'$ can be rid of control operators in the way (and the other constraints of the induction). This is the case for forms created by the rewriting rule (since they are defined by means of $\mathcal{T}$). So, for example, if the rule creates a destructor and a constructor (a redex), there is no control operator in between.

We must analize the partial evaluation, that can join edges, thus being a source of problems for the property of the lemma. Let us see the case when in the rhs we have the occurrence of a metavariable. We must discuss two types of interactions: "towards the top" and "towards the bottom".

The firing of the unique pseudo-redex in the initial rhs amounts to replace pseudo-variables with expressions of the shape $\mathbf{Abs}_n(\vec{x}.\, X)$. This reduction causes the interaction (and their erasing) of the open 0-indexed bracket on the top of $\mathbf{Abs}_n(\vec{x}.\, X)$ with the closed 0-indexed bracket in the bottom of the pseudo-variable. The brackets that face the metavariable X are also eventually deleted (apply inductive hypothesis). Now take a redex **App-Abs**. In the way there are an open 1-indexed croissant and a sequence (possibly empty) of closed 1-indexed brackets and 1-indexed fan-ins. Therefore it is possible to push them outside **Abs**. Notice that, in this way, along the auxiliary edges of the **Abs**-node, we have 1-indexed control operators.

Now, firing the pseudo-redex means that the argument of **Abs** is connected to the external environment (and this connection satisfies trivially the provisos of the induction) and the (auxiliary) arguments of the **App** are connected to the bound variables of the **Abs**. Along these last connections, we eventually have two 0-indexed brackets that face each other. Thus they can be erased.

Notice that it is not possible to create 0-indexed control nodes, besides the brackets due to the translation $\mathcal{T}$, because, according to control rules in Figure 9, 0-indexed control nodes are generated by interacting with 0-indexed croissants only. It is easy to prove that such croissants are never created.

The last check concerns the possibility of deadlock situations due to two different control operators, indexed in the same way, that face each other. Well, this is not possible, otherwise it is easy to verify that the read-back should stop there. Thus invalidating the assumptions.

The firing of other redexes can be verified in the same way. ∎

Lemma 8.10 Let N be a sharing graph having a proper redex **d-c** and whose read-back is a redex in $\mathcal{R}(N)$. Then $N \to N'$ and $\mathcal{R}(N) \longrightarrow\!\!\!\!\!\rightarrow t'$ and $t' = \mathcal{R}(N')$.

Proof (sketch): Observe that every IS-rule can be split into two reductions: a *linear* one, coinciding with the linearization step, and the firing of every pseudo-redex until the proper rhs is found. The correctness of the linear rewriting immediately follows from the static correctness of $\mathcal{T}$, the obvious fact that the access paths from the root of the rhs to the metavariables do not modify the context, and the induction hypothesis for the metavariables.

So, we must only care for the correctness of the partial evaluation, that merely concerns pseudo-forms. Note that all new paths we expect after firing a pseudo-redex in the abstract syntax tree representation of the term, are obtained by "grafting" a path from the binder to the bound variable inside a previous path traversing the application and going into some argument. By induction, we know that all these paths are obtained by reading back access paths from the sharing graph (in particular, the pseudo-variable path corresponds to a loop in the sharing graph). The correctness of this operation is guaranteed by Lemma 8.8.

Now, after firing the first pseudo-redex, we can find still pseudo-forms around. However, the shape of the expression yielded by the linearization step state clearly where these pseudo-forms are. In particular, note that every pseudo-application (except the outermost one) has a pseudo-variable as first argument (and *vice versa* every pseudo-variable is the first argument of some pseudo-application). So, after the first pseudo-reduction, we eventually find pseudo-applications connected (through access paths, as said before) with pseudo-abstractions binding metavariables. By Lemma 8.9, we can gat rid of control operators in the way, yielding pseudo-redexes. It is easy to prove, by the same reasoning as for the external pseudo-application above, that firing these pseudo-redexes we yield a graph that can be read-back into the rhs of the IS-rewriting rule. ■

Proposition 8.11 The implementation $\mathcal{T}^+$ is correct. That is, if N is a graph yielded by a derivation starting at $\mathcal{T}^+(t)$ then:

(A) $N \longrightarrow N'$ implies $\mathcal{R}(N) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{R}(N')$;

(B) N in normal form implies that also $\mathcal{R}(N)$ is in normal form.

Proof: Easy consequence of Lemmas 8.10 and 8.9. ■

9 Optimality

Optimality can be split into two tasks. The first is the existence of an effective evaluation strategy for IS's that always contracts redexes that any other evaluation strategy should eventually reduce (*call-by-need*). This is easy, since IS's are a subclass of Klop's orthogonal Combinatory Reduction Systems. Indeed, these systems own the property that the leftmost-outermost evaluation order is a call-by-need strategy [14]. The remaining task relies on showing that every redex in the sharing graph always represents a maximal family of redexes in the read-back expression. This can be yielded by switching to labeled expressions.

In particular, let us consider graphical representation of labeled expressions and labeled rewritings as described in Section 4.1. We recall from Section 5 that the initial labeled graph has edges marked by atomic labels and labels are pairwise different (property **INIT**).

We must prove that, if two redexes yielded by a labeled derivation have the same label, then they have the same representation in the sharing graph. Actually, in the discussion that follows we will allow ourselves a bit of inaccuracy, switching from labeled to unlabeled expressions and back without explicitly stating it.

As in λ-calculus (see [17]), it is possible that duplication of labels goes ahead w.r.t. the reduction of proper redexes. The problem is due to the fact that fan-nodes may duplicate labeled edges (e.g. when a fan is along a redex edge **d**-**c**). In order to cope with such situations, we must determinate, for every redex, a part of it that is never duplicated by propagation of fans. To this aim, Lamping [18] introduces the notion of *prerequisite chain* of a form **f** in the graphical representation (not in the shared graph!). Such notion is smoothly generalizable to IS's.

Definition 9.1 A *prerequisite chain* for **f** is a path in the graphical representation of an expression starting from **f** and traversing forms through principal ports, till a principal port is found. ■

So, for instance, if **d** is a destructor involved in a redex, the prerequisite chain of **d** stops at the port of the constructor (it is just an edge). In the expression $\mathbf{d}(\mathbf{d}'(\mathbf{c}))$, where **d** and **d**$'$ are destructors and **c** is a constructor, the prerequisite chain starting at **d** consists of the edge connecting **d** and **d**$'$ and the edge connecting **d**$'$ and **c**.

It is clear that the representation of a prerequisite chain in a sharing graph can never be totally duplicated by propagation of fans. Actually, a fan can not enter the chain from the ends since the principal edges of the ending forms are links of the prerequisite chain. Notice that, when a fan is in between a redex, we have two different prerequisite chains in the read-back subgraph, each corresponding to the two forms on the branches of the fan.

We want to remark that, for Lamping, prerequisite chains are not connected, since bound variables are not connected to their binders. Our graphical representation allow to overcome smartly such problem, thus yielding a simpler definition of prerequisite chain.

Theorem 9.2 Let G be a sharing graph yielded by a derivation starting at $\mathcal{T}^+(t)$ and let N be the (labeled) graphical representation of the expression read-back by G . If two prerequisite chains of the same length in N have corresponding edges with the same labels then the chains have the same representation in G and *vice versa.*

Proof: By induction over the length of the derivation. Besides the property stated by the theorem, we prove that, when a proper reduction happens, the label of the read-back redex does not appear as sub-label of any other edge. The basic case is trivial. Let us prove that the theorem is preserved by any rule of the evaluator.

The control rules preserve the paths through the nodes matched by the rule, that is the read-back is invariant w.r.t. them. Thus the theorem is obvious.

Among interfacing rules, the interesting case is when a fan traverses a form. Here the form (together with its edges) is duplicated. Let us see what happens to the prerequisite chains. There are two cases: that of a chain coming into the principal port (and stopping there) and that of a chain coming into an auxiliary port and traversing the principal edge of the form. In both cases the prerequisite chain has to traverse the fan. By induction, the two chains corresponding to the traversal of the two branches of the fan have different labels. So the duplicated chains do not represent identically labeled prerequisite chains.

The firing of a proper rule is the most difficult case. So, assume to contract a proper redex u of a sharing graph. Since the read-back of u is a prerequisite chain, by induction every edge of the set U of redexes in N in which u is read-back have the same label ℓ_u and every edge marked ℓ_u in N belongs to U. Contracting the redexes U can create new edges (that yield new prerequisite chains) and/or can copy old prerequisite chains. Notice that, by definition of the labeled reduction, every new link is marked by $(\ell_u)_s$, for some s. Furthermore, by hypothesis, there is no other occurrence of ℓ_u as sub-label of any other label in N. Therefore the labels $(\ell_u)_s$ are new w.r.t. the others.

Now take the implementation of the rhs of rewriting rules. More precisely, consider the output of the translation step. Its standard shape is $\mathbf{App}_k(M, \mathbf{Abs}_{n_1}(\vec{x}^1.X_1), \cdots, \mathbf{Abs}_{n_k}(\vec{x}^k.X_k))$. The edges that are created by the rule are exactly those in M that connect proper forms (possibly traversing control nodes only) or those connecting pseudo-forms with forms or those connecting auxiliary edges of pseudo-forms. That is, it is possible to define a bijection φ between these edges and labels $(\ell_u)_s$ created by the rule. This property is trivially preserved by the partial evaluation, since only control rules, interfacing rules or pseudo-redexes are fired. Moreover, the above bijection φ is such that, in the sharing graph yielded by the partial evaluation, an edge e is read-back into edges labeled $\varphi(e)$. This is a consequence of Lemma 8.10.

Now, let $G \to G'$ be the proper (sharing) graph reduction and $N \longrightarrow\!\!\!\!\!\rightarrow N'$ be the labeled reduction such that $\mathcal{R}(G') = N'$. We must show that, for every pair of prerequisite chains of the same length and with correspondingly equal labels, their representation in G' is the same. This is the case when the prerequisite chains have no new link, since, by Lemma 8.8 and the definition of the read-back procedure, the read-back does not change inside metavariables or outside the redex u. When the two prerequisite chains have a new link (a label of the shape $(\ell_u)_s$), according to the previous discussion, such links are actually represented by the same edge in G'. Therefore the theorem is verified. ∎

Now, notice that redexes are prerequisite chains. So the above theorem implies that redexes having the same label will be represented by the same structure in the corresponding sharing graph.

Corollary 9.3 In every derivation of the sharing graph $\mathcal{T}^+(t)$ no two redexes destructor-constructor have the same label. Therefore the graph implementation is

optimal. ■

10 Conclusions

We have generalized Lamping's optimal graph-reduction technique [18] to a new class of higher order term rewriting systems: Interaction Systems. In Interaction Systems, we may define most of the common data structures used in practice (in particular, all inductive types) and many useful constructs of real programming languages (jumps, recursions, and so on). Actually, the only constraint of IS seems to be *local sequentiality* (in the sense of Berry). For instance, the parallel or is not expressible in IS.

The main point of IS's w.r.t. optimality, is that it is particularly simple to "interface" the forms of the syntax with Lamping-Gonthier's control operators. This is a consequence of their *logical* (intuitionistic) nature, which has been deeply investigated in this paper.

Interaction Systems are a subclass of Klop's orthogonal Combinatory Reduction Systems. So the expected extension of our work is the generalization of the results described here to orthogonal Combinatory Reduction Systems. A prerequisite for fulfilling this aim is the definition of the family relation. In particular the degree of a redex becomes much more involved since the label of a tree (instead of an edge) must be taken into account.

For the same reason, we cannot hope to describe the optimal implementation of CRS in the form of an Interaction Net (since forms do not have principal ports, the decision if traversing a form with a fan cannot be local any more, but it depends from the context surrounding the form). Another problem is matching lhs of rewriting rules with the graph representing the term (if we unfold the term, we could loose sharing; if not, the sharing in the lhs must be preserved in the rhs, furtherly complicating the rewriting step). Moreover, where should a boxes be put, in the translation? Is there any linearity in CRS's? When should a box be opened? All these questions have a natural answer in IS's, due to their intuitionistic nature.

The main problem of "optimal" implementations is, however, the *efficiency!*. In particular, the accumulation of control operators (that could be exponential). During reduction, we may create sequences of control operators whose global control effect could be neglected. Thus the sequence of control operators could be safely removed. Unfortunately, this simplification can be performed only in suitable positions of the graph, without affecting Church-Rosser. Individuating these positions (and the configurations to be be reduced) does not seems to be an easy task. Some work in this direction was already done by Lamping, but his rewriting rules are not complete.

Burroni [6] and Lafont [16] have recently refined usual term rewriting systems by explicitly managing variables with control operators. The advantage is that symbolic computations can be rid of variables. In this respect, our (optimal) graph implementations is a first attempt of generalizing Burroni-Lafont's works to higher order rewriting

systems. The richer set of control operators is motivated by the presence of binding and substitution. By exploiting this analogy, one could also imagine to provide a more algebraic account of optimality.

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